SHAPE OPTIMIZATION OF POLYNOMIAL FUNCTIONALS UNDER UNCERTAINTIES ON THE RIGHT-HAND SIDE OF THE STATE EQUATION

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Abstract. The present paper is dedicated to a problem of shape optimization where the external loads applied to the structure are subject to uncertainties. The objective functional and the constraints can be written as the expected value of a polynomial functional of degree m. We provide a deterministic expression of the expectation of the polynomial as a function of the first m moments of the random variables modeling the uncertainties, as well as a method to compute its shape derivative according to the method of Hadamard. In particular, no further assumptions on the distribution of the random variables are required, and the method presented in this article is not based on computationally expensive sampling techniques. The proposed method can be applied in different contexts, like the study of the variance of a quadratic functional, or the optimization of a functional approaching the L^∞-norm of a quantity in the structure.

Key words. Shape optimization, Random right-hand side, Uncertainties, Linear elasticity, von Mises stress, polynomial functional.

MSC codes. 49Q10 , 65N75 , 65C20 , 65K10

1. Introduction.

Shape and topology optimization is a topics of ever increasing interest in the domains of engineering. The design of mechanical structures satisfying several constraints of different natures is a difficult problem for engineers, and shape optimization techniques offer an automated approach to devise original designs which satisfies with the given constraints. In the context of mechanical engineering, the optimization problem often concerns the optimization of elastic structures satisfying some requirements on their mass, and their robustness under a given mechanical load. Such robustness is usually estimated using the mechanical compliance of the structure, or by computing some yield criterion like the von Mises stress (see e.g. [6, 9]).

The increasing demand of optimized structures and the progress in computational science have resulted in the development of different optimization techniques. The approach considered in this paper relies on Hadamard's boundary variation method, treated extensively in [45, 2] and [31, chapter 5]. The profile of the structure is represented numerically by using a level-set approach on conforming meshes (see [47] for a comprehensive review of the level-set method, and [7, 4, 25, 22] for its application in the context of shape optimization). Other approaches to topology optimization include the class of density methods (see [10, 11]), among which the Solid Isotropic Material with Penalization (SIMP) method is the most widely encountered. We refer the reader to [5, 11], and to the review paper [44] for further information on the multiple techniques for shape and topology optimization. As of today, the main design softwares available on the market offer tools for structure optimization, integrating new features and developments at each release of a new version.

In industrial applications it is unrealistic to consider that all information on the problem is perfectly known. On the contrary, the presence of uncertainties on the geometry, on the material properties, and on the external loads applied on the structure must be accounted for in the design in order to assure a correct manufacturing

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process and that the performances stay acceptable under a range of uncontrolled variations of the ambient conditions. The handling of uncertainties on the shape of the domain is studied in [14, 19, 20, 21]. In [3] the authors address the issue of small uncertainties on the material properties, on the external loads, and on the geometry of the structure by linearizing the perturbation around their mean value. In [15] the mean and the variance of a generic objective functional are estimated by using a dimension reduction method followed by a Gauss-type quadrature sampling, while the shape sensitivities are computed by using the analytical derivatives of the random moments. The authors of [26] study the minimization of the mean and the variance of the mechanical compliance of an elastic structure, considering an exact expression of the random moments and their sensitivities with respect to the shape. A similar approach is adopted in [18], where the authors provide a method to compute analytically the expected value of a generic quadratic functional in terms of the first and second moments of the random variables modeling the uncertainties.

The present work adapts and extends the approach of [18] to the case of polynomial functionals of the right-hand side of the state equation. We consider the shape optimization problem as an instance of a PDE-constrained optimization problem. We suppose the right-hand side of the partial differential equation to be subject to uncertainties, without any assumption on their amplitude, and the uncertainties are modeled as random variables, by using suitable Bochner spaces. Let us consider a functional of the shape that can be expressed as a polynomial function of degree m of the solution of the state equation. Similarly to the procedure detailed in [18], we introduce a deterministic correlation tensor of order m, which depends only on the first m random moments. Consequently, it is possible to compute exactly the expected value of the functional of interest.

Our main contribution is Theorem 2.7 that provides the analytic expression of its shape derivative in terms of the first m moments of the random variables modeling the uncertainties, without any further assumption on their distributions. In the case of a finite dimensional valued uncertainty, no tensor are needed and we present in Proposition 2.8 the corresponding result. Notably, no sampling method requiring a large number of simulations is used in the method presented here.

An application of the proposed procedure is related to the utilization of the L^m-norm of a function as a smooth approximation of its L^∞-norm (i.e. its supremum) in a given domain. Indeed, by considering the L^m-norm of the stress in the domain as functional of interest, we are able to derive shapes where, on average, stress is less concentrated than in the ones obtained by controlling the expectation of the mechanical compliance.

This article is organized as follows. Section 2 states the main results of this article: it introduces the correlation operator for multilinear functionals and its applications in the context of shape optimization, with a particular focus on the context of linear elasticity. In section 3 we provide a numerical application, where a tridimensional structure is subject to an uncertain load, and its mass is minimized under a constraint on the L^6-norm of the von Mises stress in the domain. The conclusions are drawn in section 4. For ease of reading, we recall in Appendix A the mathematical structures and techniques required to state the shape optimization problem and its solution. In particular, we recall the definition and properties of the Hadamard derivative, introduce the notion of tensor product between Hilbert spaces and projective product space, and sketch a model for dealing with uncertainties. Finally, the needed details on the numerical implementation for the reproduction of the results are gathered in Appendix B.

2. Main results.

2.1. Correlation operator and multilinear functionals. We begin our study by introducing the correlation operator for multilinear functionals defined on Hilbert spaces under uncertainties. The random component of the problem is treated by considering functions defined on suitable Bochner spaces. More details on the proerties of Bochner spaces and tensor products are provided in Appendix A. The correlation operator has been studied in the context of shape optimization under uncertainties in [18], limitedly to bilinear functionals defined on Hilbert spaces.

Let us consider the measure space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$, where $\mathcal{O}$ is the event space, $\mathcal{F} \subset 2^{\mathcal{O}}$ is a σ-algebra on $\mathcal{O}$, and $\mathbb{P}$ is a probability measure, and let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space. First of all, we can state a result about the Bochner-integrability of the tensor product.

PROPOSITION 2.1. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_m$ be Hilbert spaces, each endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_i}$ for $i = 1, \ldots, m$. Let us consider $x_1, \ldots, x_m$, each belonging to the Bochner space $\mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{H}_i)$. Finally, we define the mapping $\omega \mapsto \bigotimes_{i=1}^m x_i(\omega)$ from the event space $\mathcal{O}$ to the Hilbert space $\bigotimes_{i=1}^m \mathcal{H}_i$. Then, such a function belongs to the Bochner space $\mathrm{L}^1(\mathcal{O}, \mathbb{P}; \bigotimes_{i=1}^m \mathcal{H}_i)$.*

Proof. We consider the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_m$ as well as their tensor product $\bigotimes_{i=1}^m \mathcal{H}_i$ as Banach spaces with respect to the norms induced by their respective inner products. In order to prove that $\bigotimes_{i=1}^m x_i(\cdot) \in \mathrm{L}^1(\mathcal{O}, \mathbb{P}; \bigotimes_{i=1}^m \mathcal{H}_i)$, we estimate its norm, and we use Hölder's inequality extended to multiple terms

$$\begin{aligned}\int_{\mathcal{O}} \left\| \bigotimes_{i=1}^m x_i(\omega) \right\|_{\bigotimes_{i=1}^m \mathcal{H}_i} \mathrm{d}\mathbb{P}(\omega) &= \int_{\mathcal{O}} \left(\prod_{i=1}^m \|x_i(\omega)\|_{\mathcal{H}_i} \right) \mathrm{d}\mathbb{P}(\omega) \\ &\leq \prod_{i=1}^m \left(\int_{\mathcal{O}} \|x_i(\omega)\|_{\mathcal{H}_i}^m \right)^{\frac{1}{m}} \mathrm{d}\mathbb{P}(\omega) = \prod_{i=1}^m \|x_i\|_{\mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{H}_i)} < \infty. \quad \square\end{aligned}$$

Next, the correlation operator is introduced. As it is remarked in [18], the literature is not consistent in the definition of the correlation between random variables. In this paper, we adopt the following definition.

DEFINITION 2.2 (Correlation operator on Bochner spaces). *Let $\left(\mathcal{H}_i, \langle \cdot, \cdot \rangle_{\mathcal{H}_i}\right)$ be Hilbert spaces for $i = 1, \ldots, m$. Let us consider the linear operator defined on $\prod_{i=1}^m \mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{H}_i)$, mapping $(x_1(\cdot), \ldots, x_m(\cdot))$ to $\bigotimes_{i=1}^m x_i(\omega)$. Thanks to Proposition 2.1, we know that the function $\omega \mapsto \bigotimes_{i=1}^m x_i(\omega)$ is Bochner-integrable.*

The correlation between the m functions $x_1, \ldots, x_m$ is defined as

$$\mathrm{Cor}(x_1, \ldots, x_m) = \mathbb{E}\left[\bigotimes_{i=1}^m x_i(\omega) \right] \in \bigotimes_{i=1}^m \mathcal{H}_i,$$

and the correlation operator $\mathrm{Cor} : \prod_{i=1}^m \mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{H}_i) \to \bigotimes_{i=1}^m \mathcal{H}_i$ is a bounded linear operator associating to m random vectors their correlation. If all arguments of the correlation operator are the same, we denote $\mathrm{Cor}_m(x) = \mathrm{Cor}(x, \ldots, x)$.

In [43, 42] the term $\mathrm{Cor}_m(x)$ is denoted as the stochastic moment of order m of x.

Finally, we state a proposition that allows the expression of the expected value of a multilinear expression in terms of a correlation tensor.

PROPOSITION 2.3. *Let* $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ *be a probability space,* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *Hilbert spaces provided with the norms* $\|\cdot\|_{\mathcal{H}_i}$ *for* $i = 1 \ldots m$*, and* $P : \prod_{i=1}^m \mathcal{H}_i \to \mathbb{R}$ *a bounded multilinear operator. Then, there exists a unique bounded, real-valued, linear operator* $\widehat{P}_m$ *defined on* $\bigotimes_{i=1}^m \mathcal{H}_i$ *such that the following three statements hold true for all* $(x_1, \ldots, x_m) \in \prod_{i=1}^m \mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{H}_i)$*:*

1. $P(x_1, \ldots, x_m) \in \mathrm{L}^1(\mathcal{O}, \mathbb{P})$*,*
2. $P(x_1(\omega), \ldots, x_m(\omega)) = \widehat{P}_m\left(\bigotimes_{i=1}^m x_i(\omega)\right)$*, for almost all* $\omega \in \mathcal{O}$*,*
3. $\mathbb{E}\left[P(x_1, \ldots, x_m)\right] = \widehat{P}_m\left(\mathrm{Cor}\left(x_1, \ldots, x_m\right)\right)$*.*

Proof. The first point comes directly from the continuity of the operator P and the application of Hölder's inequality. The second can be deduced from the universal property of the tensor product (see [13, Chapter 9], [34, Theorem 2.6.4], and Proposition A.6).

In order to prove the third statement, we show that the hypotheses of [32, Proposition 1.2.3] (reported in Appendix A as Proposition A.12) are verified. The function $\omega \mapsto P\left(x_1(\omega), \ldots, x_m(\omega)\right)$ is Bochner-integrable thanks to the first point of this proposition. The operator $\widehat{P}_m$ is continuous, as proved by the second point of this proposition. Therefore, we can apply [32, Proposition 1.2.3] and conclude:

$$\mathbb{E}\left[P(x_1, \ldots, x_m)\right] = \mathbb{E}\left[\widehat{P}_m\left(\bigotimes_{i=1}^m x_i\right)\right] = \widehat{P}_m\left(\bigotimes_{i=1}^m x_i\right) = \widehat{P}_m\left(\mathrm{Cor}\left(x_1, \ldots, x_m\right)\right).$$

□

2.2. Uncertain loads in linear elasticity. Often, before computing the value of the objective functional in an optimization problem, it is necessary to pass through an intermediate step that is the computation of the state of the system. For any $\Omega \in \mathcal{S}_{adm}$, let $\mathcal{X}_\Omega$ and Y be Hilbert spaces, and $\mathcal{A}_\Omega : \mathcal{X}_\Omega \to Y$ a bounded, linear, invertible functional. Let us consider the following optimization problem:

$$\left|\begin{array}{l}\text{Find } \Omega_{\text{opt}} \in \mathcal{S}_{adm} \text{ minimizing } \Omega \mapsto \mathbb{E}\left[\mathcal{J}\left(\Omega, \mathbf{g}\right)\right] = \mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right], \\ \text{where } \mathcal{A}_\Omega\, \mathbf{u}(\omega) = \mathbf{g}(\omega) \quad \text{almost surely,}\end{array}\right. \tag{2.1}$$

where $P_\Omega : \prod_{i=1}^m \mathcal{X}_\Omega \to \mathbb{R}$ is a bounded m-multilinear functional. The term $\mathbf{u}$ is said to be the state of the system. The state equation $\mathcal{A}_\Omega\, \mathbf{u} = \mathbf{g}$ can be interpreted as a constraint in the optimization problem, and might require the solution of a partial differential equation. We remark that, since $\mathbf{g} \in \mathrm{L}^m(\mathcal{O}, \mathbb{P}; Y)$ is a random process, $\mathbf{u}$ is a random process in the Bochner space $\mathrm{L}^m(\mathcal{O}, \mathbb{P}; \mathcal{X}_\Omega)$ thanks to the usual elliptic *a priori* estimates.

From now on, we focus on shape optimization problems in the context of linear elasticity. Further information on the theory of linear elasticity can be found in [46, 33].

DEFINITION 2.4 (Strain and stress tensors). *Let us consider the two Lamé coefficients* λ *and* μ *such that the quantities* μ*, and* $2\mu + d\lambda$ *are strictly positive. For any* $\mathbf{v} \in \mathrm{H}^1\left(\mathbb{R}^d\right)^d$*, representing a displacement field, the infinitesimal strain tensor* $\boldsymbol{\epsilon}(v)$ *is defined as* $\boldsymbol{\epsilon}(v) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2}$*. The Cauchy stress tensor* $\boldsymbol{\sigma}(\mathbf{v})$ *is defined as the following linear application of the strain tensor, according to Hooke's law:*

$$\boldsymbol{\sigma}(\mathbf{v}) = \mathbf{A}\nabla \mathbf{v} = 2\mu\boldsymbol{\epsilon}(\mathbf{v}) + \lambda(\mathrm{div}\ \mathbf{v}). \tag{2.2}$$

where $\mathbf{A}$ *is the fourth-order stiffness tensor.*

We consider the following shape optimization problem, particular case of problem (2.1) for linear elasticity.

$$
\left|
\begin{array}{l}
\text{Find } \Omega_{\mathrm{opt}} \in \mathcal{S}_{adm} \text{ minimizing } \Omega \mapsto \mathbb{E}\left[\mathcal{J}\left(\Omega, \mathbf{g}\right)\right] = \mathbb{E}\left[P_{\Omega}(\mathbf{u}, \cdots, \mathbf{u})\right], \\
\text{where, almost surely, the state } \mathbf{u} \text{ solves:} \\
\qquad \left\{
\begin{array}{rcll}
-\mathrm{div}\, \mathbf{A}\nabla\mathbf{u} & = & \mathbf{0} & \text{in } \Omega, \\
\mathbf{A}\nabla\mathbf{u}\,\mathbf{n} & = & \mathbf{g} & \text{on } \Gamma_{\mathrm{N}}, \\
\mathbf{A}\nabla\mathbf{u}\,\mathbf{n} & = & \mathbf{0} & \text{on } \Gamma_{0}, \\
\mathbf{u} & = & \mathbf{0} & \text{on } \Gamma_{\mathrm{D}}.
\end{array}
\right.
\end{array}
\right. \tag{2.3}
$$

In problem (2.3) we denote Γ_{N}, Γ_0 and Γ_{D} three open disjoint portions of the border of Ω with strictly positive measure. If we consider the mechanical load applied on Γ_{N} to be a random variable $\mathbf{g} \in \mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathrm{L}^2\left(\Gamma_{\mathrm{N}}\right)\right)$, we can conclude that $\mathbf{u}(\omega) \in \mathrm{H}^1\left(\Omega\right)^d$ for almost all event ω, and $\mathbf{u} \in \mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathrm{H}^1\left(\Omega\right)^d\right)$.

Denoting $\gamma : \mathrm{H}^1\left(\Omega\right)^d \to \mathrm{L}^2\left(\Gamma_{\mathrm{N}}\right)$ the operator mapping $\mathbf{v} \mapsto \mathbf{v}|_{\Gamma_{\mathrm{N}}}$, the problem defining the state equation can be written in variational form:

$$
\left|
\begin{array}{l}
\text{Find } \mathbf{u}(\omega) \in \mathcal{V} = \mathrm{H}^1_{\Gamma_{\mathrm{D}}}\left(\Omega\right)^d \text{ such that for all } \mathbf{v} \in \mathcal{V}: \\
\qquad \left\langle \mathbf{A}\nabla\mathbf{u}(\omega), \nabla\mathbf{v}\right\rangle_{\mathrm{L}^2(\Omega)} = \left\langle \gamma(\mathbf{v}), \mathbf{g}(\omega)\right\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}, \\
\text{where} \\
\qquad \left\langle \mathbf{A}\nabla\mathbf{u}(\omega), \nabla\mathbf{v}\right\rangle_{\mathrm{L}^2(\Omega)} = \displaystyle\int_{\Omega} \left(\mathbf{A}\nabla\mathbf{u}(\omega) : \nabla\mathbf{v}\right) \mathrm{d}\mathbf{x}, \\
\qquad \left\langle \gamma(\mathbf{v}), \mathbf{g}(\omega)\right\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{N}})} = \displaystyle\int_{\Gamma_{\mathrm{N}}} \mathbf{g}(\omega) \cdot \mathbf{v}\, \mathrm{d}\mathrm{s}.
\end{array}
\right. \tag{2.4}
$$

For simplicity, we suppose that all admissible shapes in $\mathcal{S}_{adm}$ share the portions Γ_{N} and Γ_{D}, constraining the displacements fields $\boldsymbol{\theta} \in \Theta_{adm} \subset \mathrm{W}^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ to be equal to $\mathbf{0}$ on these surfaces. Moreover, we focus our study on functionals P_{Ω} with the following structure:

$$
P_{\Omega}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \int_{\Omega} q_0(\mathbf{v}_1(\mathbf{x}), \ldots, \mathbf{v}_m(\mathbf{x}))\, \mathrm{d}\mathbf{x} + \int_{\Omega} q_1(\nabla\mathbf{v}_1(\mathbf{x}), \ldots, \nabla\mathbf{v}_m(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{2.5}
$$

where $q_0 : \prod_{i=1}^m \mathbb{R}^d \to \mathbb{R}$ and $q_1 : \prod_{i=1}^m \mathbb{R}^{d\times d} \to \mathbb{R}$ are multilinear and continuous.

Without any further assumption on the domain, problem (2.3) can be affected by regularity issues (see [38, 29, 12]). Indeed, the Lax-Milgram theorem ensures that, for almost any $\omega \in \mathcal{O}$, $\mathbf{u}(\omega) \in \mathrm{H}^1\left(\Omega\right)^d$. However, for $P_{\Omega}(\mathbf{u}(\omega), \ldots, \mathbf{u}(\omega))$ to be well-defined we must require that $\mathbf{u} \in \mathrm{W}^{1,m}\left(\Omega\right)$. In order to verify such condition we consider that, for all admissible domain $\Omega \in \mathcal{S}_{adm}$, the portions of the boundary where Dirichlet and Neumann conditions are applied are fully separated as

$$
\overline{\Gamma_{\mathrm{D}}} \cap \overline{\left(\Gamma_{\mathrm{N}} \cup \Gamma_0\right)} = \emptyset. \tag{2.6}
$$

We recall the following result on the regularity of the solution of boundary value problems.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^{k+2}$, with k integer. We suppose that its boundary can be divided in three parts Γ_{D}, Γ_{N}, and Γ_0*

mutually disjoint, with strictly positive measure, satisfying the condition (2.6)*. Let us consider* $\mathbf{g} \in \mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})^d$*. Then, the solution* $\mathbf{u}$ *of problem (2.4) belongs to the Sobolev space* $\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)$*, and there exists* $C > 0$ *such that the following estimate holds:*

$$\|\mathbf{u}\|_{\mathrm{H}^{k+2}(\Omega)} \leq C \|\mathbf{g}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})}. \tag{2.7}$$

Proof. At first, we remark that $\widetilde{\mathbf{g}}$, extension of $\mathbf{g}$ to $\Gamma_\mathrm{N} \cup \Gamma_0$ such that $\widetilde{\mathbf{g}}|_{\Gamma_0} = 0$ belongs to $\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N} \cup \Gamma_0)^d$. Moreover, $\|\mathbf{g}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})} = \|\widetilde{\mathbf{g}}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N}\cup\Gamma_0)}$.

Under the hypothesis (2.6) the elliptic regularity estimates apply on the entire domain Ω. In particular, there exists some constant $\widetilde{C} > 0$ such that (see [41, Theorem 8.29])

$$\|\mathbf{u}\|_{\mathrm{H}^{k+2}(\Omega)} \leq \widetilde{C} \left(\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)} + \|\widetilde{\mathbf{g}}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N}\cup\Gamma_0)} \right). \tag{2.8}$$

By the coercivity of the bilinear form and the Poincaré and trace inequalities, there exists three positive constants α, C_1, C_2 such that

$$\begin{aligned} \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} &\leq \frac{1}{\alpha} \langle \mathbf{A}\nabla\mathbf{u}, \nabla\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = \langle \gamma(\mathbf{u}), \mathbf{g}\rangle_{\mathrm{L}^2(\Gamma_\mathrm{N})} \\ &\leq C_1 \|\mathbf{u}\|_{\mathrm{L}^2(\Gamma_\mathrm{N})} \|\widetilde{\mathbf{g}}\|_{\mathrm{L}^2(\Gamma_\mathrm{N}\cup\Gamma_0)} \leq C_2 \|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} \|\widetilde{\mathbf{g}}\|_{\mathrm{L}^2(\Gamma_\mathrm{N}\cup\Gamma_0)}. \end{aligned}$$

Injecting this result into (2.8) we conclude that

$$\begin{aligned} \|\mathbf{u}\|_{\mathrm{H}^{k+2}(\Omega)} &\leq \widetilde{C} \left(\|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} + \|\widetilde{\mathbf{g}}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N}\cup\Gamma_0)} \right) \\ &\leq \widetilde{C} \left(C_2 \|\widetilde{\mathbf{g}}\|_{\mathrm{L}^2(\Gamma_\mathrm{N}\cup\Gamma_0)} + \|\widetilde{\mathbf{g}}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N}\cup\Gamma_0)} \right) \leq C \|\widetilde{\mathbf{g}}\|_{\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N}\cup\Gamma_0)} \end{aligned}$$

for some positive constant C. ∎

The following result details the computation of the shape derivative of $P_\Omega(\mathbf{u}, \dots, \mathbf{u})$. We remark that, even if we only need $\mathbf{u}$ to be in $\mathrm{W}^{1,m}_{\Gamma_\mathrm{D}}(\Omega)^d$ to define $P_\Omega(\mathbf{u}, \dots, \mathbf{u}(\omega))$, we require the higher regularity in $\mathrm{W}^{1,2m-2}_{\Gamma_\mathrm{D}}(\Omega)^d$ to compute its derivative.

Proposition 2.6. *Let* k *be a positive integer such that* $k + 1 > d/2$*. Let* $\Omega \in \mathcal{S}_{adm}$ *be a bounded domain with a* $\mathcal{C}^{k+2}$ *boundary,* $\mathbf{g} \in \mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})$*, and* $\mathbf{u}$ *solution of problem (2.4). We suppose that* $P_\Omega : \prod_{i=1}^m \mathrm{W}^{1,m}_{\Gamma_\mathrm{D}}(\Omega)^d \to \mathbb{R}$ *has the structure presented in* (2.5)*. Then,* $\mathbf{u} \in \mathrm{W}^{1,2m-2}_{\Gamma_\mathrm{D}}(\Omega)$*, and the quantity* $P_\Omega(\mathbf{u}, \dots, \mathbf{u})$ *is differentiable with respect to* Ω*. The adjoint state* $\mathbf{w}$ *solving the following boundary-value problem is well-defined in* $\mathrm{H}^1(\Omega)$

$$\left\{ \begin{array}{rcll} -\mathrm{div}\, \mathbf{A}\nabla\mathbf{w} &=& \displaystyle\sum_{i=1}^m \partial_i q_0(\mathbf{u}, \dots, \mathbf{u}) - \mathrm{div}\, \partial_i q_1(\nabla\mathbf{u}, \dots, \nabla\mathbf{u}) & \textit{in } \Omega, \\ \Big(\mathbf{A}\nabla\mathbf{w}\Big)\mathbf{n} &=& \displaystyle\sum_{i=1}^m \partial_i q_1(\nabla\mathbf{u}, \dots, \nabla\mathbf{u})\mathbf{n} & \textit{on } \Gamma_\mathrm{N} \cup \Gamma_0, \\ \mathbf{w} &=& \mathbf{0} & \textit{on } \Gamma_\mathrm{D}. \end{array} \right. \tag{2.9}$$

Finally, the shape derivative of $P_\Omega(\mathbf{u},\dots,\mathbf{u})$ can be expressed as follows

$$\begin{aligned}(2.10)\quad \frac{\mathrm{d}}{\mathrm{d}\Omega}P_\Omega(\mathbf{u},\dots,\mathbf{u})(\boldsymbol{\theta}) &= \int_{\Gamma_0}\Big(q_0(\mathbf{u}(\mathbf{s}),\dots,\mathbf{u}(\mathbf{s}))\Big)(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\\ &+\int_{\Gamma_0}\Big(q_1(\nabla\mathbf{u}(\mathbf{s}),\dots,\nabla\mathbf{u}(\mathbf{s}))\Big)(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}-\int_{\Gamma_0}\Big(\mathbf{A}\nabla\mathbf{u}(\mathbf{s}):\nabla\mathbf{w}(\mathbf{s})\Big)(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}.\end{aligned}$$

Proof. At first, we prove the regularity of $\mathbf{u}$ to ensure that $P_\Omega(\mathbf{u},\dots,\mathbf{u})$ is well-defined. The displacement $\mathbf{u}$ solves the elliptic boundary-value problem (2.4). Given the regularity $\mathcal{C}^{k+2}$ of the domain and the fact that $\mathbf{g}\in \mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})$, we can apply Proposition 2.5 and prove that $\mathbf{u}\in \mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)$. The domain Ω is bounded and of class $\mathcal{C}^{k+2}$, thus it complies with the cone condition defined as in [1, Definition 4.6]. Thanks to the Sobolev embedding theorem (see [1, Theorem 4.12, part I]), the space $\mathrm{H}^{k+2}(\Omega)$ is compactly embedded into $\mathrm{W}^{1,m}(\Omega)$ for any $\widetilde{p}>2$. Thus, we can conclude that $\mathbf{u}\in \mathrm{W}^{1,2m-2}_{\Gamma_\mathrm{D}}(\Omega)\subset \mathrm{W}^{1,m}_{\Gamma_\mathrm{D}}(\Omega)$.

The shape derivative and the adjoint problem can be computed by the fast derivation method developed by Céa (see [17] and [2, Section 6.4.3]). We introduce a Lagrangian function $\mathcal{L}:\mathcal{S}_{adm}\times \mathrm{H}^{k+2}\left(\mathbb{R}^d\right)^d\times \mathrm{H}^1\left(\mathbb{R}^d\right)^d\to\mathbb{R}$ such that

$$\begin{aligned}\mathcal{L}(\Omega;\hat{\mathbf{u}};\hat{\mathbf{w}}) &= \int_\Omega q_0(\hat{\mathbf{u}},\dots,\hat{\mathbf{u}})\,\mathrm{d}\mathbf{x}+\int_\Omega q_1(\nabla\hat{\mathbf{u}},\dots,\nabla\hat{\mathbf{u}})\,\mathrm{d}\mathbf{x}-\int_\Omega \mathbf{A}\nabla\hat{\mathbf{u}}:\nabla\hat{\mathbf{w}}\,\mathrm{d}\mathbf{x}\\ &+\int_{\Gamma_\mathrm{N}}\mathbf{g}\cdot\gamma(\hat{\mathbf{w}})\,\mathrm{d}\mathbf{s}+\int_{\Gamma_\mathrm{D}}\left(\gamma(\hat{\mathbf{w}})\cdot\big(\mathbf{A}\nabla\hat{\mathbf{u}}\big)\mathbf{n}+\gamma(\hat{\mathbf{u}})\cdot\Big(\mathbf{A}\nabla\hat{\mathbf{w}}-\sum_{j=1}^N\partial_i q_1(\nabla\hat{\mathbf{u}},\dots,\nabla\hat{\mathbf{u}})\Big)\mathbf{n}\right)\,\mathrm{d}\mathbf{s}.\end{aligned}$$

All arguments of the Lagrangian are independent, since $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are defined on the whole space $\mathbb{R}^d$, and not only on Ω. The term defined as an integral on the portion Γ_D of the boundary enforces the Dirichlet boundary condition, similarly to the proof of [7, Theorem 7]. The partial derivative $\frac{\partial\mathcal{L}}{\partial\hat{w}}(\Omega,\hat{\mathbf{u}},\hat{\mathbf{w}})$ vanishes when evaluated in $\hat{\mathbf{u}}=\mathbf{u}$ thanks to the definition of the state equation (2.4). Thus, for any $\hat{\mathbf{w}}\in \mathrm{H}^1\left(\mathbb{R}^d\right)^d$ we have that

$$(2.11)\quad \mathcal{L}(\Omega;\mathbf{u};\hat{\mathbf{w}})=\int_\Omega q_0(\mathbf{u}(\mathbf{x}),\dots,\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}+\int_\Omega q_1(\nabla\mathbf{u}(\mathbf{x}),\dots,\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}=P_\Omega(\mathbf{u},\dots,\mathbf{u}).$$

Let us focus on the adjoint state $\mathbf{w}$ solution of (2.9). The weak form of (2.9) can be written as follows

$$(2.12)\quad \left|\begin{array}{l}\text{Find } \mathbf{w}\in\mathcal{V}=\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)^d \text{ such that}\\ \text{for all } \mathbf{v}\in\mathcal{V}:\\ \qquad \langle\mathbf{A}\nabla\mathbf{w},\nabla\mathbf{v}\rangle_{\mathrm{L}^2(\Omega)}=\displaystyle\sum_{i=1}^m\int_\Omega\partial_i q_0(\mathbf{u},\dots,\mathbf{u})\mathbf{v}\,\mathrm{d}\mathbf{x}+\sum_{i=1}^m\int_\Omega\partial_i q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u})\nabla\mathbf{v}\,\mathrm{d}\mathbf{x}.\end{array}\right.$$

The well-posedness of problem (2.12) is proven by the Lax-Milgram theorem. The bilinear form is the classical elasticity operator, which is continuous and coercive. The left-hand side is continuous by the definition of the operator P_Ω and by the regularity of $\mathbf{u}$. Indeed, since q_0 and q_1 are m-multilinear continuous operators (see (2.5)), there exist two positive constants C_0 and C_1 such that

$$\begin{aligned} |q_0(\mathbf{y}_1, \dots, \mathbf{y}_m)| &\leq C_0 \prod_{j=1}^m \|\mathbf{y}_j\|_{\mathbb{R}^d} && \text{for any} \quad \mathbf{y}_1, \dots, \mathbf{y}_m \in \mathbb{R}^d, \\ |q_1(\mathbf{Y}_1, \dots, \mathbf{Y}_m)| &\leq C_1 \prod_{j=1}^m \|\mathbf{Y}_j\|_{\mathbb{R}^{d\times d}} && \text{for any} \quad \mathbf{Y}_1, \dots, \mathbf{Y}_m \in \mathbb{R}^{d\times d}. \end{aligned}$$

Thus

$$\begin{aligned} &\left| \sum_{i=1}^m \int_\Omega \partial_i q_0(\mathbf{u}, \dots, \mathbf{u}) \mathbf{v} \,\mathrm{d}\mathbf{x} + \sum_{i=1}^m \int_\Omega \partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u}) \nabla \mathbf{v} \,\mathrm{d}\mathbf{x} \right| \\ &\quad \leq \sum_{i=1}^m \int_\Omega |\partial_i q_0(\mathbf{u}, \dots, \mathbf{u}) \mathbf{v}| \,\mathrm{d}\mathbf{x} + \sum_{i=1}^m \int_\Omega |\partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u}) \nabla \mathbf{v}| \,\mathrm{d}\mathbf{x} \\ &\quad \leq m \left(C_0 \int_\Omega \|\mathbf{u}(\mathbf{x})\|_{\mathbb{R}^d}^{m-1} \|\mathbf{v}(\mathbf{x})\|_{\mathbb{R}^d} \,\mathrm{d}\mathbf{x} + C_1 \int_\Omega \|\nabla \mathbf{u}(\mathbf{x})\|_{\mathbb{R}^{d\times d}}^{m-1} \|\nabla \mathbf{v}(\mathbf{x})\|_{\mathbb{R}^{d\times d}} \,\mathrm{d}\mathbf{x} \right) \\ &\quad \leq m\, C_0 \|\mathbf{v}\|_{\mathrm{L}^2(\Omega)} \|\mathbf{u}\|_{\mathrm{L}^{2m-2}(\Omega)^{m-1}} + m\, C_1 \|\nabla \mathbf{v}\|_{\mathrm{L}^2(\Omega)} \|\nabla \mathbf{u}\|_{\mathrm{L}^{2m-2}(\Omega)^{m-1}} \\ &\quad \leq \left(m(C_0 + C_1) \|\mathbf{u}\|_{\mathrm{W}^{1,2m-2}(\Omega)}^{m-1} \right) \|\mathbf{v}\|_{\mathrm{H}^1(\Omega)} . \end{aligned}$$

Having proved that the adjoint problem (2.9) is well-posed, we remark that the derivative $\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{u}}}(\Omega, \mathbf{u}, \hat{\mathbf{w}})$ vanishes when evaluated in $\hat{\mathbf{w}} = \mathbf{w}$. Indeed

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{u}}}(\Omega, \mathbf{u}, \mathbf{w})(\mathbf{v}) = -\int_\Omega (\mathbf{A}\nabla \mathbf{w} : \nabla \mathbf{v}) \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{D}}} \gamma(\mathbf{v}) \big(\mathbf{A}\nabla \mathbf{w}\mathbf{n}\big) \,\mathrm{d}\mathbf{s} \\ &- \sum_{i=1}^m \left(\int_{\Gamma_{\mathrm{D}}} \gamma(\mathbf{v}) \cdot \Big(\partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u})\Big) \mathbf{n} \,\mathrm{d}\mathbf{s} + \int_\Omega \big(\partial_i q_0(\mathbf{u}, \dots, \mathbf{u})(\mathbf{v}) + \partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u})(\nabla \mathbf{v})\big) \,\mathrm{d}\mathbf{x} \right) \\ &= \int_\Omega \mathbf{v} \cdot \Big(\mathrm{div}\ \mathbf{A}\nabla \mathbf{w}\Big) \,\mathrm{d}\mathbf{x} + \sum_{i=1}^m \int_\Omega \mathbf{v} \cdot \Big(\partial_i q_0(\mathbf{u}, \dots, \mathbf{u}) - \mathrm{div}\ \partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u})\Big) \,\mathrm{d}\mathbf{x} \\ &- \int_{\Gamma_{\mathrm{N}} \cup \Gamma_0} \gamma(\mathbf{v}) \cdot \Big(\mathbf{A}\nabla \mathbf{w} - \sum_{i=1}^m \partial_i q_1(\nabla \mathbf{u}, \dots, \nabla \mathbf{u})\Big) \mathbf{n} \,\mathrm{d}\mathbf{s} + \int_{\Gamma_{\mathrm{D}}} \gamma(\mathbf{v}) \cdot \Big(\mathbf{A}\nabla \mathbf{w}\mathbf{n}\Big) \,\mathrm{d}\mathbf{s} = 0. \end{aligned}$$

We conclude computing the expression (2.10) of the shape derivative of $P_\Omega(\mathbf{u}, \dots, \mathbf{u})$. Recalling (2.11) and the results of [31, Section 5] on shape differentiation we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\Omega} P_\Omega(\mathbf{u}, \dots, \mathbf{u})(\boldsymbol{\theta}) = \frac{\mathrm{d}}{\mathrm{d}\Omega} \mathcal{L}\,(\Omega; \mathbf{u}; \mathbf{w})(\boldsymbol{\theta}) \\ &\quad = \frac{\partial}{\partial \Omega} \mathcal{L}\,(\Omega; \mathbf{u}; \mathbf{w})(\boldsymbol{\theta}) + \frac{\partial}{\partial \hat{\mathbf{u}}} \mathcal{L}\,(\Omega; \mathbf{u}; \mathbf{w}) \frac{\mathrm{d}}{\mathrm{d}\Omega} \mathbf{u}(\boldsymbol{\theta}) + \frac{\partial}{\partial \hat{\mathbf{w}}} \mathcal{L}\,(\Omega; \mathbf{u}; \mathbf{w}) \frac{\mathrm{d}}{\mathrm{d}\Omega} \mathbf{w}(\boldsymbol{\theta}) \\ &\quad = \int_{\Gamma_0} \Big(q_0(\mathbf{u}(\mathbf{s}), \dots, \mathbf{u}(\mathbf{s})) + q_1(\nabla \mathbf{u}(\mathbf{s}), \dots, \nabla \mathbf{u}(\mathbf{s})) - \mathbf{A}\nabla \mathbf{u}(\mathbf{s}) : \nabla \mathbf{w}(\mathbf{s}) \Big) (\boldsymbol{\theta} \cdot \mathbf{n}) \,\mathrm{d}\mathbf{s} \square \end{aligned}$$

We have proven the well-posedness of the variational problems and the expression of the shape derivative in the deterministic case. Thus, if we consider the applied load $\mathbf{g}$ to be a random variable belonging to the Bochner space $\mathrm{L}^m \left(\mathcal{O}, \mathbb{P}; \mathrm{H}^{k+\frac{1}{2}}\,(\Gamma_{\mathrm{N}})\right)$, the results of Proposition 2.6 apply for almost any event $\omega \in \mathcal{O}$.

Thanks to the framework adopted in [42, Theorem 3.2], we introduce the following tensorialized bounded linear operators, defined on Hilbert spaces:

$$
\begin{array}{llll}
\widehat{\mathbf{A}}_m & : \left(\mathrm{L}^2(\Omega)^d\right)^{\otimes m} \to \left(\mathrm{L}^2(\Omega)^d\right)^{\otimes m} & \text{s.t.} \quad \widehat{\mathbf{A}}_m \bigotimes_{i=1}^m \mathbf{V}_i & \mapsto \bigotimes_{i=1}^m (\mathbf{A}\mathbf{V}_i), \\
\widehat{\nabla}_m & : \left(\mathrm{H}^1(\Omega)^d\right)^{\otimes m} \to \left(\mathrm{L}^2(\Omega)\right)^{\otimes m} & \text{s.t.} \quad \widehat{\nabla}_m \bigotimes_{i=1}^m \mathbf{v}_i & \mapsto \bigotimes_{i=1}^m (\nabla \mathbf{v}_i), \\
\widehat{\gamma}_m & : \left(\mathrm{H}^1(\Omega)^d\right)^{\otimes m} \to \left(\mathrm{H}^{-1/2}(\Gamma_\mathrm{N})^d\right)^{\otimes m} & \text{s.t.} \quad \widehat{\gamma}_m \bigotimes_{i=1}^m \mathbf{v}_i & \mapsto \bigotimes_{i=1}^m \mathbf{v}_i|_{\Gamma_\mathrm{N}}.
\end{array}
$$

Using the tensor notation we can state the following result concerning the computation of the objective of problem (2.3) and its shape derivative.

THEOREM 2.7. *Let us consider $\mathcal{S}_{adm}$ to be a class of regular enough admissible shapes sharing the portions Γ_D and Γ_N of their boundaries, so that any $\Omega \in \mathcal{S}_{adm}$ is of class $\mathcal{C}^{k+2}$ with $k+1 > \frac{dm}{2}$ integer. Moreover, set $\mathbf{g} \in \mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})\right)$. If $\mathbf{u}$ solves problem (2.4) almost surely, then:*

(i) $\mathbf{u}$ *belongs to* $\mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$.

(ii) $\mathrm{Cor}_m(\mathbf{u}) \in \left(\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)^d\right)^{\otimes m}$ *is solution of the following problem*

(2.13)
$$
\left|
\begin{array}{l}
\textit{Find } \mathrm{Cor}_m(\mathbf{u}) \in \mathcal{V} = \left(\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)^d\right)^{\otimes m} \\
\textit{such that, for all } V \in \mathcal{V}: \\
\left\langle \widehat{\mathbf{A}}_m \widehat{\nabla}_m \mathrm{Cor}_m(\mathbf{u}), \widehat{\nabla}_m V \right\rangle_{(\mathrm{L}^2(\Omega))^{\otimes m}} = \langle \widehat{\gamma}_m(V), \mathrm{Cor}_m(\mathbf{g}) \rangle_{(\mathrm{L}^2(\Gamma_\mathrm{N}))^{\otimes m}}.
\end{array}
\right.
$$

Moreover, $\mathrm{Cor}_m(\mathbf{u})$ *belongs to* $\bigotimes_{i=1}^m \mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d$.

(iii) Let $\widehat{P_{\Omega}}_m$ be the tensorization of the operator P_Ω on $\left(\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)\right)^{\otimes m}$. We denote $\widehat{q_0}_m : (\mathbb{R}^d)^{\otimes m} \to \mathbb{R}$ and $\widehat{q_1}_m : (\mathbb{R}^{d\times d})^{\otimes m} \to \mathbb{R}$ the tensorizations of q_0 and q_1 respectively. Then $\mathbb{E}\left[P_\Omega(\mathbf{u}, \ldots, \mathbf{u})\right] = \widehat{P_{\Omega}}_m(\mathrm{Cor}_m(\mathbf{u}))$, with

$$
\widehat{P_{\Omega}}_m(\mathrm{Cor}_m(\mathbf{u})) = \int_\Omega \left(\widehat{q_0}_m + \widehat{q_1}_m \widehat{\nabla}_m\right)\left(\mathrm{Cor}_m(\mathbf{u})\right)(\mathbf{x}, \ldots, \mathbf{x})\mathrm{d}\mathbf{x}. \tag{2.14}
$$

(iv) The shape derivative of $\mathbb{E}\left[P_\Omega(\mathbf{u}, \ldots, \mathbf{u})\right]$ can be written as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[P_\Omega(\mathbf{u}, \ldots, \mathbf{u})\right](\boldsymbol{\theta}) = & \int_{\Gamma_0} \left(\widehat{q_0}_m + \widehat{q_1}_m \widehat{\nabla}_m\right)\left(\mathrm{Cor}_m(\mathbf{u})\right)(\mathbf{s}, \ldots, \mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\mathrm{d}\mathbf{s} \\
& - \int_{\Gamma_0} \langle \mathbf{A}\nabla, \nabla \rangle \left(\mathrm{Cor}(\mathbf{u}, \mathbf{w})\right)(\mathbf{s}, \mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\mathrm{d}\mathbf{s},
\end{aligned} \tag{2.15}
$$

where the mapping $\langle \mathbf{A}\nabla, \nabla \rangle : \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) \otimes \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) \to \mathrm{L}^1(\Omega)$ is induced from the bilinear form $(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \mapsto \mathbf{A}\nabla\hat{\mathbf{u}} : \nabla\hat{\mathbf{w}}$. The term $\mathrm{Cor}(\mathbf{u}, \mathbf{w})$ solves the adjoint *problem problem*

$$
(2.16)\quad \left|\begin{array}{l}
\textit{Find } \operatorname{Cor}(\mathbf{u},\mathbf{w}) \in \mathcal{W} = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)^d \otimes \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)^d \\
\textit{such that, for all } W \in \mathcal{W}: \\
\left\langle \Big(\mathbf{A}\nabla \otimes \mathbf{A}\nabla\Big) \operatorname{Cor}(\mathbf{u},\mathbf{w}), \Big(\nabla\otimes\nabla\Big) W \right\rangle_{\mathrm{L}^2(\Omega)^d \otimes \mathrm{L}^2(\Omega)^d} \\
\qquad = \displaystyle\sum_{i=1}^m \left\langle (\gamma\otimes\mathbb{I})\, W, \operatorname{Cor}(\mathbf{g}, \partial_i q_0(\mathbf{u},\dots,\mathbf{u}))\right\rangle_{\mathrm{L}^2(\Gamma_\mathrm{N})^d\otimes \mathrm{L}^2(\Omega)^d} \\
\qquad + \displaystyle\sum_{i=1}^m \left\langle (\gamma\otimes\nabla)\, W, \operatorname{Cor}(\mathbf{g}, \partial_i q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u}))\right\rangle_{\mathrm{L}^2(\Gamma_\mathrm{N})^d\otimes \mathrm{L}^2(\Omega)^d}.
\end{array}\right.
$$

Proof. In order to prove (i) we recall that the estimate (2.7) on the norm of $\mathbf{u}$ holds for almost any $\omega \in \mathcal{O}$. Thus, the solution of (2.4) belongs to the space $\mathrm{H}^{k+2}(\Omega)$ almost surely. Moreover, [42, Theorem 2.1] assures that, since $\mathbf{g} \in \mathrm{L}^2\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+1/2}(\Gamma_\mathrm{N})\right)$, the random solution $\mathbf{u}$ of problem (2.4) is unique and belongs to $\mathrm{L}^2\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$. Therefore, by the uniqueness of $\mathbf{u}$ and the elliptical estimates, we can state that $\mathbf{u} \in \mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$. Point (ii) can be proven by [42, Theorem 3.2], which assures the well-posedness of problem (2.13) and the uniqueness of the solution in $\mathcal{V}$. In order to prove the regularity of $\operatorname{Cor}_m(\mathbf{u})$ we use point (i) and Proposition 2.1 to show that, since $\mathbf{u}$ belongs to the Bochner space $\mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$, then

$$
\operatorname{Cor}_m(\mathbf{u}) = \mathbb{E}\left[(\mathbf{u})^{\otimes m}\right] \in \left(\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)^{\otimes m}.
$$

The existence of the linear continuous operator $\widehat{P_\Omega}_m : \left(\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)^{\otimes m} \to \mathbb{R}$ in point (iii) is a direct application of Proposition 2.3. Indeed, since $\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega) \subset \mathrm{W}^{1,m}_{\Gamma_\mathrm{D}}(\Omega)$ by the Sobolev embedding theorem, the restriction of the m-multilinear functional P_Ω to $\prod_{i=1}^m \mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)$ is well-defined. Hence, Proposition 2.3 ensures the existence and uniqueness of the linear operator $\widehat{P_\Omega}_m$ such that $\mathbb{E}[P_\Omega(\mathbf{u},\dots,\mathbf{u})] = \widehat{P_\Omega}_m(\operatorname{Cor}_m(\mathbf{u}))$.

The expression (2.14) for $\mathbb{E}[P_\Omega(\mathbf{u},\dots,\mathbf{u})]$ derives from the linearity of the expectation operator

$$
\begin{aligned}
\mathbb{E}[P_\Omega(\mathbf{u},\dots,\mathbf{u})] &= \mathbb{E}[P_\Omega(\mathbf{u},\dots,\mathbf{u})] \\
&= \int_\Omega \mathbb{E}[q_0(\mathbf{u}(\mathbf{x}),\dots,\mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbb{E}[q_1(\nabla\mathbf{u}(\mathbf{x}),\dots,\nabla\mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \left(\widehat{q_0}_m\Big(\operatorname{Cor}_m(\mathbf{u})(\mathbf{x},\dots,\mathbf{x})\Big) + \widehat{q_1}_m\Big(\widehat{\nabla}_m \operatorname{Cor}_m(\mathbf{u})(\mathbf{x},\dots,\mathbf{x})\Big)\right)\mathrm{d}\mathbf{x} \\
&= \int_\Omega \left(\widehat{q_0}_m + \widehat{q_1}_m \widehat{\nabla}_m\right)\Big(\operatorname{Cor}_m(\mathbf{u})\Big)(\mathbf{x},\dots,\mathbf{x})\,\mathrm{d}\mathbf{x} = \widehat{P_\Omega}_m(\operatorname{Cor}_m(\mathbf{u})).
\end{aligned}
$$

The variational formulation (2.16) for $\operatorname{Cor}(\mathbf{u},\mathbf{w})$ in point (iv) can be deduced from the application of [42, Theorem 3.2] to $\mathbf{u}\otimes\mathbf{w}$, knowing that, for almost any $\omega \in \mathcal{O}$, $\mathbf{u}(\omega)$ solves problem (2.4) and $\mathbf{w}(\omega)$ solves problem (2.12). We remark that, by point (i), $\mathbf{u}$ belongs to $\mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$ and, by Proposition 2.6, also to

$\mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{W}^{1,2m-2}_{\Gamma_\mathrm{D}}(\Omega)^d\right)$. Thus

$$\sum_{i=1}^m\left(\partial_i q_0(\mathbf{u},\dots,\mathbf{u})+\partial_i q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u})\right)\in\mathrm{L}^{1-\frac{1}{m}}\left(\mathcal{O},\mathbb{P};\mathrm{L}^2(\Omega)\right).$$

Since $\mathbf{g}\in\mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})\right)\subset\mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{L}^2(\Gamma_\mathrm{N})\right)$ and thanks to Proposition 2.1 we have that

$$\mathbf{g}\otimes\sum_{i=1}^m\left(\partial_i q_0(\mathbf{u},\dots,\mathbf{u})+\partial_i q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u})\right)\in\mathrm{L}^1\left(\mathcal{O},\mathbb{P};\mathrm{L}^2(\Gamma_\mathrm{N})\otimes\mathrm{L}^2(\Omega)\right).$$

Therefore

$$\mathrm{Cor}\left(\mathbf{g},\sum_{i=1}^m\left(\partial_i q_0(\mathbf{u},\dots,\mathbf{u})+\partial_i q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u})\right)\right)\in\mathrm{L}^2(\Gamma_\mathrm{N})\otimes\mathrm{L}^2(\Omega)$$

and the right-hand side of the variational formulation in (2.16) is continuous, assuring the well-posedness of problem (2.16) by the Lax-Milgram theorem.

In order to retrieve the expression (2.15) of the shape derivative of $\mathbb{E}\left[P_\Omega(\mathbf{u},\dots,\mathbf{u})\right]$, we consider the derivative of the objective function for a fixed event $\omega\in\mathcal{O}$ found in the expression (2.10) of Proposition 2.6. For each event $\omega\in\mathcal{O}$ we introduce the adjoint state $\mathbf{w}(\omega)$ as the solution of problem (2.12). The expression of the derivative of $\mathbb{E}\left[P_\Omega(\mathbf{u},\dots,\mathbf{u})\right]$ can be found computing the expectation of Proposition 2.6 and applying the tensorized operators introduced earlier

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[P_\Omega(\mathbf{u},\dots,\mathbf{u})\right](\boldsymbol{\theta})=&\,\mathbb{E}\left[\int_{\Gamma_0}\Big(q_0(\mathbf{u},\dots,\mathbf{u})\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\right]\\
&+\mathbb{E}\left[\int_{\Gamma_0}\Big(q_1(\nabla\mathbf{u},\dots,\nabla\mathbf{u})\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}-\int_{\Gamma_0}\Big(\mathbf{A}\nabla\mathbf{u}:\nabla\mathbf{w}\Big)(\mathbf{s},\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\right]\\
=&\,\mathbb{E}\left[\int_{\Gamma_0}\Big(\widehat{q_0}_m(\mathbf{u})^{\otimes m}\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}+\int_{\Gamma_0}\Big(\widehat{q_1}_m\widehat{\nabla}_m(\mathbf{u})^{\otimes m}\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\right]\\
&-\mathbb{E}\left[\int_{\Gamma_0}\Big(\langle\mathbf{A}\nabla,\nabla\rangle(\mathbf{u}\otimes\mathbf{w})\Big)(\mathbf{s},\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\right]\\
=&\int_{\Gamma_0}\widehat{q_0}_m\Big(\mathrm{Cor}_m(\mathbf{u})\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\mathrm{d}\mathbf{s}+\int_{\Gamma_0}\widehat{q_1}_m\widehat{\nabla}_m\Big(\mathrm{Cor}_m(\mathbf{u})\Big)(\mathbf{s},\dots,\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\\
&-\int_{\Gamma_0}\langle\mathbf{A}\nabla,\nabla\rangle\Big(\mathrm{Cor}(\mathbf{u},\mathbf{w})\Big)(\mathbf{s},\mathbf{s})(\boldsymbol{\theta}\cdot\mathbf{n})\,\mathrm{d}\mathbf{s}\qquad\square
\end{aligned}$$

2.3. Shape derivatives under finite-rank noise. For this section, we consider k to be a positive integer such that $k+1\geq d/2$. We consider that $\mathbf{g}$ can be written in terms of a finite number of random variables $X_1,\dots,X_m\in\mathrm{L}^m(\mathcal{O},\mathcal{F},\mathbb{P})$ and loads $\mathbf{g}_1,\dots,\mathbf{g}_N\in\mathrm{H}^{k+\frac{1}{2}}(\Gamma_\mathrm{N})$ as

$$\mathbf{g}(\omega)=\sum_{j=1}^N X_j(\omega)\mathbf{g}_j. \tag{2.17}$$

We introduce the following notations:

- $\mathcal{A}_{(1,m),N} = \{1,\ldots,N\}^m$ is the set of all m-tuples whose elements are integers between 1 and N;
- $\mathcal{A}^{i,j}_{(1,m),N} = \left\{\vec{\mathbf{k}} \in \mathcal{A}_{(1,m),N} \text{ such that } k_i = j\right\} \subset \mathcal{A}_{(1,m),N}$ is the subset of all m-tuples in $\mathcal{A}_{(1,m),N}$ whose i-th element is equal to j;
- we denote $C^j_{\vec{\mathbf{k}}}$ the number of times the integer j appears in the m-tuple $\vec{\mathbf{k}}$
- finally, we denote $\alpha(\vec{\mathbf{k}})$ the following quantity:

$$\alpha(\vec{\mathbf{k}}) = \alpha(k_1,\ldots,k_m) = \prod_{j=1}^{N}\left(\mathbb{E}\left[X_j^{C^j_{\vec{\mathbf{k}}}}\right]\right).$$

PROPOSITION 2.8. *Let $\Omega \in \mathcal{S}_{adm}$ be a $\mathcal{C}^{k+2}$ domain, and P_Ω be an m-multilinear continuous functional following the structure (2.5). Moreover, let $\mathbf{g} \in \mathrm{L}^m\left(\mathcal{O},\mathbb{P};\mathrm{H}^{k+\frac{1}{2}}(\Gamma_{\mathrm{N}})\right)$ be a random mechanical load such that it can be decomposed as in (2.17), where the N real random variables $X_i \in \mathrm{L}^m(\mathcal{O},\mathcal{F},\mathbb{P})$ are mutually independent, and let $\mathbf{u}_j \in \mathrm{H}^{k+2}_{\Gamma_{\mathrm{D}}}(\Omega)$ be the solution of the elasticity equation under the load $\mathbf{g}_j$ for $j \in \{1,\ldots,N\}$.*

Then, $\mathbb{E}[P_\Omega(\mathbf{u},\ldots,\mathbf{u})]$ can be written as

$$\mathbb{E}[P_\Omega(\mathbf{u},\ldots,\mathbf{u})] = \sum_{\vec{\mathbf{k}}\in\mathcal{A}_{(1,m),N}}\left(\alpha(\vec{\mathbf{k}})\int_\Omega \left(q_0(\mathbf{u}_{k_1},\ldots,\mathbf{u}_{k_m}) + q_1(\nabla\mathbf{u}_{k_1},\ldots,\nabla\mathbf{u}_{k_m})\right)\mathrm{d}\mathbf{x}\right) \tag{2.18}$$

Furthermore, we can write its the shape derivative in Ω as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}[P_\Omega(\mathbf{u},\ldots,\mathbf{u})](\boldsymbol{\theta}) = &-\sum_{j=1}^{N}\int_{\Gamma_0}(\boldsymbol{\theta}\cdot\mathbf{n})\,(\mathbf{A}\nabla\mathbf{u}_j : \nabla\mathbf{w}_j)\,\mathrm{d}\mathbf{s}\\ &+\sum_{\vec{\mathbf{k}}\in\mathcal{A}_{(1,m),N}}\alpha(\vec{\mathbf{k}})\left(\int_{\Gamma_0}(\boldsymbol{\theta}\cdot\mathbf{n})\,\left(q_0(\mathbf{u}_{k_1},\ldots,\mathbf{u}_{k_m}) + q_1(\nabla\mathbf{u}_{k_1},\ldots,\nabla\mathbf{u}_{k_m})\right)\mathrm{d}\mathbf{s}\right)\end{aligned} \tag{2.19}$$

where the N states $\mathbf{u}_1,\ldots,\mathbf{u}_N$ solve the state equation for $\mathbf{g}_1,\ldots,\mathbf{g}_N$ respectively and belong to $\mathrm{H}^{k+2}_{\Gamma_{\mathrm{D}}}(\Omega)$, while the N adjoint states $\mathbf{w}_1,\ldots,\mathbf{w}_N$ belong to $\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and solve the following adjoint problems

$$\begin{cases} -\operatorname{div}\mathbf{A}\nabla\mathbf{w}_j = \displaystyle\sum_{i=1}^{m}\sum_{\vec{\mathbf{k}}\in\mathcal{A}^{i,j}_{(1,m),N}}\alpha(\vec{\mathbf{k}})\Big(\partial_i q_0(\mathbf{u}_{k_1},\ldots,\mathbf{u}_{k_m}) - \operatorname{div}\partial_i q_1(\nabla\mathbf{u}_{k_1},\ldots,\nabla\mathbf{u}_{k_m})\Big) & \text{in } \Omega,\\ (\mathbf{A}\nabla\mathbf{w}_j)\,\mathbf{n} = \displaystyle\sum_{i=1}^{m}\sum_{\vec{\mathbf{k}}\in\mathcal{A}^{i,j}_{(1,m),N}}\alpha(\vec{\mathbf{k}})\left(\partial_i q_1(\nabla\mathbf{u}_{k_1},\ldots,\nabla\mathbf{u}_{k_m})\right)^T\mathbf{n} & \text{on } \Gamma_0\cup\Gamma_{\mathrm{N}},\\ \mathbf{w}_j = \mathbf{0} & \text{on } \Gamma_{\mathrm{D}}. \end{cases} \tag{2.20}$$

Proof. At first we remark that, by the linearity of the elasticity equation, the decomposition (2.17) can be extended to the displacement $\mathbf{u}$ as

$$\mathbf{u}(\omega) = \sum_{j=1}^{N} X_j(\omega)\mathbf{u}_j$$

The expression (2.19) derives directly from the linearity of the expected value and the

m-linearity of P_Ω. Indeed, for almost all event $\omega \in \mathcal{O}$,

$$P_\Omega\left(\mathbf{u}(\omega), \ldots, \mathbf{u}(\omega)\right) = \sum_{\vec{\mathbf{k}} \in \mathcal{A}_{(1,m),N}} \left(\left(\prod_{j=1}^{N} X_j^{C^j_{\vec{\mathbf{k}}}} \right) P_\Omega(\mathbf{u}_{k_1}(\omega), \ldots, \mathbf{u}_{k_m}(\omega)) \right).$$

Therefore

$$\mathbb{E}\left[P_\Omega\left(\mathbf{u}, \ldots, \mathbf{u}\right)\right] = \sum_{\vec{\mathbf{k}} \in \mathcal{A}_{(1,m),N}} \alpha(\vec{\mathbf{k}}) P_\Omega(\mathbf{u}_{k_1}, \ldots, \mathbf{u}_{k_m}).$$

In order to compute the shape derivative of $\mathbb{E}\left[P_\Omega\left(\mathbf{u}, \ldots, \mathbf{u}\right)\right]$ we use once again Céa's fast derivative method [17, 2] as done for Proposition 2.6. We introduce the following Lagrangian function $\mathcal{L} : \mathcal{S}_{adm} \times (\mathrm{H}^1\left(\mathbb{R}^d\right)^d)^N \times (\mathrm{H}^1\left(\mathbb{R}^d\right)^d)^N \to \mathbb{R}$ associated to problem (2.3) where the state equation is seen as a PDE constraint

(2.21)

$$\begin{aligned}
\mathcal{L}\left(\Omega; \hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_N; \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N\right) = -\sum_{j=1}^{N} \Big\{ \int_\Omega \left(\mathbf{A}\nabla\hat{\mathbf{u}}_j : \nabla\hat{\mathbf{w}}_j\right) \mathrm{d}\mathbf{x} - \int_{\Gamma_\mathrm{N}} \mathbf{g}_j \cdot \hat{\mathbf{w}}_j \, \mathrm{d}\mathbf{s} \\
- \int_{\Gamma_\mathrm{D}} \left(\hat{\mathbf{w}}_j \cdot \left(\mathbf{A}\nabla\hat{\mathbf{u}}_j\right)\mathbf{n} + \hat{\mathbf{u}}_j \cdot \left(\mathbf{A}\nabla\hat{\mathbf{w}}_j - \sum_{\ell}^{N} \partial_\ell q_1(\nabla\hat{\mathbf{u}}_{k_1}, \ldots, \nabla\hat{\mathbf{u}}_{k_m}) \right)\mathbf{n} \right) \mathrm{d}\mathbf{s} \Big\} \\
+ \sum_{\vec{\mathbf{k}} \in \mathcal{A}_{(1,m),N}} \Big\{ \alpha(\vec{\mathbf{k}}) \big(\int_\Omega q_0(\hat{\mathbf{u}}_{k_1}, \ldots, \hat{\mathbf{u}}_{k_m}) \, \mathrm{d}\mathbf{x} + \int_\Omega q_1(\nabla\hat{\mathbf{u}}_{k_1}, \ldots, \nabla\hat{\mathbf{u}}_{k_m}) \, \mathrm{d}\mathbf{x} \big) \Big\}
\end{aligned}$$

The variables $\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N$ act as Lagrange multipliers for the PDE constraints of the terms $\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_N$. In order to assure that all arguments of the Lagrangian are independent, the terms $\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_N$ and $\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N$ are defined on the whole space $\mathbb{R}^d$, and not only on Ω.

By construction, the terms $\mathbf{u}_1, \ldots, \mathbf{u}_N$ solving the equation $\frac{\partial\mathcal{L}}{\partial\hat{\mathbf{w}}_j} = 0$, are also solutions of the state equation for the right-hand side $\mathbf{g} = \mathbf{g}_j$. Thus, we can express the functional $\mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right]$ in terms of the Lagrangian:

$$\mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right] = \mathcal{L}\left(\Omega; \mathbf{u}_1, \ldots, \mathbf{u}_N; \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N\right), \tag{2.22}$$

for all $\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N \in \mathrm{H}^1\left(\mathbb{R}^d\right)$.

The expression for the shape derivative of the functional of interest is found differentiating equation (2.22) with respect to the domain Ω

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right](\boldsymbol{\theta}) = \frac{\mathrm{d}}{\mathrm{d}\Omega}\mathcal{L}\left(\Omega; \mathbf{u}_1, \ldots, \mathbf{u}_N; \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N\right)(\boldsymbol{\theta}) \\
&= \frac{\partial\mathcal{L}}{\partial\Omega}\left(\Omega, \mathbf{u}_1, \ldots, \mathbf{u}_N, \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N\right)(\boldsymbol{\theta}) + \sum_{j=1}^{N} \frac{\partial\mathcal{L}}{\partial\hat{\mathbf{u}}_i}\left(\Omega, \mathbf{u}_1, \ldots, \mathbf{u}_N, \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N\right)(\mathbf{u}_j').
\end{aligned} \tag{2.23}$$

The term $\mathbf{u}_j'$ denotes the Eulerian derivative of $\mathbf{u}_j$, which is defined as the derivative of the mapping $t \mapsto \mathbf{u}_j(\Omega_{t\boldsymbol{\theta}})$ in $t = 0$, where $\mathbf{u}_j(\Omega_{t\boldsymbol{\theta}})$ is the unique solution of the state equation for the right-hand side $\mathbf{g}_j$ and on the deformed domain $\Omega_{t\boldsymbol{\theta}}$.

Next, we remark that, by choosing $\mathbf{w}_1, \dots, \mathbf{w}_N$ solving the adjoint problem (2.20), the quantity $\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{u}}_j}$ vanishes for $j = 1 \dots N$. Indeed, for any $\mathbf{v} \in \mathrm{H}^1\left(\mathbb{R}^d\right)$ we have

$$
\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{u}}_j}(\Omega, \mathbf{u}_1, \dots, \mathbf{u}_N, \mathbf{w}_1, \dots, \mathbf{w}_N)(\mathbf{v}) \\
&= \sum_{i=1}^{m} \sum_{\vec{\mathbf{k}} \in \mathcal{A}^{i,j}_{(1,m),N}} \int_\Omega \alpha(\vec{\mathbf{k}}) \big(\partial_i q_0(\mathbf{u}_{k_1}, \dots, \mathbf{u}_{k_N})(\mathbf{v}) + \partial_i q_1(\nabla \mathbf{u}_{k_1}, \dots, \nabla \mathbf{u}_{k_N}) : (\nabla \mathbf{v}) \big) \, \mathrm{d}\mathbf{x} \\
&\quad - \int_\Omega (\mathbf{A} \nabla \mathbf{w}_j : \nabla \mathbf{v}) \, \mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{D}}} (\mathbf{w}_j \cdot (\mathbf{A} \nabla \mathbf{v} \mathbf{n}) + \mathbf{v} \cdot (\mathbf{A} \nabla \mathbf{w}_j \mathbf{n})) \, \mathrm{d}\mathbf{s} \\
&= \int_\Omega \left(\sum_{i=1}^{m} \sum_{\vec{\mathbf{k}} \in \mathcal{A}^{i,j}_{(1,m),N}} \alpha(\vec{\mathbf{k}}) \big(\partial_i q_0(\mathbf{u}_1, \dots, \mathbf{u}_N) - (\mathrm{div}\ \partial_i q_1(\nabla \mathbf{u}_{k_1}, \dots, \nabla \mathbf{u}_{k_N})) \big) + \mathrm{div}\ \mathbf{w}_j \right) \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \\
&\quad + \int_{\Gamma_{\mathrm{N}}} \left(-\mathbf{A} \nabla \mathbf{w}_j \, \mathbf{n} + (\partial_i q_1(\nabla \mathbf{u}_{k_1}, \dots, \nabla \mathbf{u}_{k_N}))^T \, \mathbf{n} \right) \cdot \mathbf{v} \, \mathrm{d}\mathbf{s} + \int_{\Gamma_{\mathrm{D}}} \mathbf{w}_j \, (\mathbf{A} \nabla \mathbf{v} \, \mathbf{n}) \, \mathrm{d}\mathbf{s} = 0.
\end{aligned}
$$

By taking $\mathbf{w}_1, \dots, \mathbf{w}_N$ as solutions of problem (2.20) for $j = 1 \dots N$, we can further simplify the expression (2.23) for the shape derivative of $\mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right]$ and obtain

$$
\frac{\mathrm{d}}{\mathrm{d}\Omega} \mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right](\boldsymbol{\theta}) == \frac{\partial \mathcal{L}}{\partial \Omega} \left(\Omega, \mathbf{u}_1, \dots, \mathbf{u}_N, \mathbf{w}_1, \dots, \mathbf{w}_N\right)(\boldsymbol{\theta}). \tag{2.24}
$$

For simplicity, we consider the portions Γ_{N} and Γ_{D} of the boundary to be non-optimizable, which is equivalent to narrow the set of admissible displacement fields $\boldsymbol{\theta}$ to the set Θ_{adm} defined as

$$
\Theta_{adm} = \left\{ \boldsymbol{\theta} \in \mathrm{W}^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \, : \, \boldsymbol{\theta} = 0 \text{ on } \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} \right\}.
$$

Thanks to the restriction of the admissible displacement fields to Θ_{adm} and to [31, Theorem 5.2.2], we conclude that the shape derivative of $\mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right]$ can be expressed as

$$
\frac{\mathrm{d}}{\mathrm{d}\Omega} \mathbb{E}\left[P_\Omega(\mathbf{u}, \cdots, \mathbf{u})\right](\boldsymbol{\theta}) = \frac{\partial \mathcal{L}}{\partial \Omega} \left(\Omega, \mathbf{u}_1, \dots, \mathbf{u}_N, \mathbf{w}_1, \dots, \mathbf{w}_N\right)(\boldsymbol{\theta}) = \int_{\Gamma_0} (\boldsymbol{\theta} \cdot \mathbf{n}) \mathcal{A}(\mathbf{s}) \, \mathrm{d}\mathbf{s}.
$$

with

$$
\mathcal{A} = -\sum_{i=1}^{m} (\mathbf{A} \nabla \mathbf{u}_i : \nabla \mathbf{w}_i) + \sum_{\vec{\mathbf{k}} \in \mathcal{A}_{(1,m),N}} \alpha(\vec{\mathbf{k}}) \big(q_0(\mathbf{u}_{k_1}, \dots, \mathbf{u}_{k_m}) + q_1(\nabla \mathbf{u}_{k_1}, \dots, \nabla \mathbf{u}_{k_m}) \big).
$$

□

It is worth remarking that the method presented in this section requires the computation of only N adjoint states. Moreover, the PDEs defining the states $\mathbf{u}_1, \dots, \mathbf{u}_N$ and the adjoint states $\mathbf{w}_1, \dots, \mathbf{w}_N$ all share the structure of their left-hand side. This property can be very useful for the numerical simulations since, by inverting once the matrix representing the discretization of the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega (\mathbf{A} \nabla \mathbf{u} : \nabla \mathbf{v}) \, \mathrm{d}\mathbf{x}$, we can solve the $2N$ boundary value problems faster.

Let us denote $\mathcal{N}\left(P_\Omega, N\right)$ the minimal number of terms to be computed in (2.18) and (2.19) to express the expected value of the functional and its derivative. In the most general case, $\mathcal{N}(P_\Omega, N) = N^m$, since we have to compute all the terms in the form $P_\Omega(\mathbf{g}_{k_1}, \ldots, \mathbf{g}_{k_m})$, as well as their shape derivatives. However, this number can be reduced if the multilinear functional P_Ω shows some symmetries among its arguments. Indeed, if P_Ω is completely symmetric, we have $\mathcal{N}(P_\Omega, N) = \binom{N+m-1}{m}$.

3. Application: structural optimization under constraints on the von Mises stress.

3.1. Estimate of the expected value of the von Mises stress.

An application of polynomial functionals in shape optimization is related to the approximation of the L^∞-norm of a given quantity in a structure by the L^m-norm, for m sufficiently large. A significant concern in structural mechanics is the design of structures where the stress is as evenly distributed as possible, preventing stress concentrations that could compromise the integrity of the component. This requirement suggests the use of functionals with order $m > 2$ in order to better penalize stress concentrations than quadratic functionals.

As a showcase, we study the optimization of a 3D linear elastic structure (thus $d = 3$) with respect to its volume and the L^m-norm of the von Mises stress, for $m \geq 2$ even integer. We suppose that the optimization problem is framed as problem (2.3), and that the random external load $\mathbf{g} \in \mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathrm{L}^2(\Omega)\right)$ can be decomposed as in (2.17), with $k \geq 1$ integer.

We introduce the von Mises stress as reported in [33, Section 4.5.6] and in [9].

Definition 3.1 (Deviatoric tensors and von Mises stress). *In each point of the domain $\Omega \subset \mathbb{R}^3$, we define the deviatoric component of the strain and stress tensors as*

$$\boldsymbol{\epsilon}_{\mathrm{VM}}(\mathbf{u}) = \boldsymbol{\epsilon}(\mathbf{u}) - \frac{1}{d}\mathbb{I}\,\mathrm{tr}\,\boldsymbol{\epsilon}(\mathbf{u}) = \boldsymbol{\epsilon}(\mathbf{u}) - \frac{1}{d}\mathbb{I}\,\mathrm{div}\;\mathbf{u}$$
$$\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}) = \boldsymbol{\sigma}(\mathbf{u}) - \frac{1}{d}\mathbb{I}\,\mathrm{tr}\,\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\,\boldsymbol{\epsilon}(\mathbf{u}) - \frac{2\mu}{d}\,\mathrm{tr}\,\boldsymbol{\epsilon}(\mathbf{u}) = 2\mu\,\boldsymbol{\epsilon}_{\mathrm{VM}}(\mathbf{u}).$$

The von Mises stress is defined in each point of the domain as:

$$s_{\mathrm{VM}}(\mathbf{u}) = \sqrt{\frac{d}{2}\,\left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u})\right)}.$$

We are interested in estimating the expected value of the functional

$$\Omega \mapsto \mathcal{G}_m(\Omega, \mathbf{g}) = G(\mathbf{u}, \ldots, \mathbf{u}), \tag{3.1}$$

where $G : \mathrm{W}^{1,m}(\Omega)^3 \to \mathbb{R}$ is such that

$$G(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \int_\Omega \left(\left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{v}_1) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{v}_2)\right) \ldots \left(\left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{v}_{m-1}) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{v}_m)\right)\right)\,\mathrm{d}\mathbf{x}. \tag{3.2}$$

At first, we can observe that, for a given displacement field $\mathbf{u} \in \mathrm{H}^{k+2}(\Omega) \subset \mathrm{W}^{1,m}(\Omega)$, the quantity $G(\mathbf{u}, \ldots, \mathbf{u})$ is equal to the L^m-norm of the von Mises stress $s_{\mathrm{VM}}(\mathbf{u})$ in Ω, elevated to the power m

$$G(\mathbf{u}, \ldots, \mathbf{u}) = \left(\int_\Omega |s_{\mathrm{VM}}(\mathbf{u})|^m\;\mathrm{d}\mathbf{x}\right) = \|s_{\mathrm{VM}}(\mathbf{u})\|^m_{\mathrm{L}^m(\Omega)}.$$

Moreover, because of the concavity of the mapping $x \mapsto \sqrt[m]{x}$, the following bound on the expectation of the L^m-norm of the von Mises stress holds

$$\mathbb{E}\left[\|s_{\mathrm{VM}}(\mathbf{u})\|_{\mathrm{L}^m(\Omega)}\right] \leq \left(\mathbb{E}\left[G(\mathbf{u},\dots,\mathbf{u})\right]\right)^{\frac{1}{m}}. \tag{3.3}$$

Finally, we remark that the functional G respects the structure defined in (2.5). Therefore, we can apply Proposition 2.8 to compute the shape derivative of the functional $\Omega \mapsto \mathcal{G}_m(\Omega,\mathbf{g})$. The expression of the functional can be further simplified by considering the symmetries between the arguments of $\mathcal{G}_m$.

Definition 3.2. *We establish the following notation.*

- *We denote*

$$\mathcal{B}_{\ell,N} = \left\{ \vec{\boldsymbol{\rho}} \in \mathbb{N}^{N\times N} : 0 \leq \rho_{ij} \leq \ell \text{ and } \sum_{i,j=1}^{N} \rho_{ij} = \ell \right\}$$

  the set of all $N\times N$ integer matrices whose entries are positive and their sum is equal to ℓ. The cardinality of said set can be computed as $|\mathcal{B}_{\ell,N}| = \binom{N^2+\ell-1}{\ell}$.
- *For ℓ and N positive integers and $\vec{\boldsymbol{\rho}} \in \mathcal{B}_{\ell,N}$, we define the following multinomial coefficient*

$$\binom{\ell}{\vec{\boldsymbol{\rho}}} = \frac{\ell!}{\prod_{i,j=1}^{N}(\rho_{i,j}!)}.$$

- *For N real random variables $X_1,\dots,X_m \in \mathrm{L}^m(\mathcal{O},\mathbb{P};\mathbb{R})$ and $\vec{\boldsymbol{\rho}} \in \mathcal{B}_{\frac{m}{2},N}$, we denote:*

$$K(\vec{\boldsymbol{\rho}}) = \binom{\frac{m}{2}}{\vec{\boldsymbol{\rho}}} \prod_{j=1}^{N} \mathbb{E}\left[X_j^{\sum_{k=1}^{N}(\rho_{kj}+\rho_{jk})}\right].$$

Having introduced the necessary notation to take the symmetries among the arguments into account, we can write the expectation of the functional $\mathcal{G}_m(\Omega,\mathbf{g})$ as

$$\mathbb{E}\left[\mathcal{G}_m(\Omega,\mathbf{g})\right] = \sum_{\vec{\boldsymbol{\rho}}\in\mathcal{B}_{\frac{m}{2},N}} \left\{ K(\vec{\boldsymbol{\rho}}) \int_\Omega \prod_{j,k=1}^{N} \left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}_j) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}_k)\right)^{\rho_{jk}} \,\mathrm{d}\mathbf{x} \right\}, \tag{3.4}$$

where each $\mathbf{u}_j$ solves the state equation problem (2.4) with the loadings $\mathbf{g}_j$ for $j \in \{1,\dots,N\}$.

Since the functional G respects the structure defined in (2.5), we can apply Proposition 2.8 and find the following expression for the shape derivative of $\mathbb{E}\left[\mathcal{G}_m(\Omega,\mathbf{g})\right]$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[\mathcal{G}_m(\Omega,\mathbf{g})\right](\boldsymbol{\theta}) &= \frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[G(\mathbf{u},\dots,\mathbf{u})\right] \\ &= \int_{\Gamma_0} (\boldsymbol{\theta}\cdot\mathbf{n}) \left(-\sum_{j=1}^{N} (\mathbf{A}\nabla\mathbf{u}_j : \nabla\mathbf{w}_j) + \sum_{\vec{\boldsymbol{\rho}}\in\mathcal{B}_{\frac{m}{2},N}} \left\{ K(\vec{\boldsymbol{\rho}}) \prod_{j,k=1}^{N} \left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}_j) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathbf{u}_k)\right)^{\rho_{jk}} \right\} \right) \mathrm{d}s.\end{aligned} \tag{3.5}$$

The adjoint states $\mathbf{w}_1, \ldots, \mathbf{w}_N$ solve the following adjoint equations

$$\left\{\begin{array}{rcll} -\mathrm{div}\ \mathbf{A}\nabla\mathbf{w}_j & = & -2\mu\,\mathrm{div}\left(\displaystyle\sum_{k=1}^{N} L_{jk}\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_k\right)\right) & \text{in } \Omega \\ \left(\mathbf{A}\nabla\mathbf{w}_j\right)\mathbf{n} & = & 2\mu\left(\displaystyle\sum_{k=1}^{N} L_{jk}\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_k\right)\right)\mathbf{n} & \text{on } \Gamma_0 \cup \Gamma_{\mathrm{N}} \\ \mathbf{w}_j & = & \mathbf{0} & \text{on } \Gamma_{\mathrm{D}}, \end{array}\right. \tag{3.6}$$

where the terms $L_{jk} \in \mathrm{L}^{m-1}(\Omega)$ are defined as

$$L_{jk} = 2\sum_{\vec{\boldsymbol{\rho}}\in\mathcal{B}_{\frac{m}{2},N}}\left(K(\vec{\boldsymbol{\rho}})\rho_{jk}\left(\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_j\right):\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_k\right)\right)^{\rho_{jk}-1}\prod_{\ell\neq k}\left(\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_j\right):\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathbf{u}_\ell\right)\right)^{\rho_{j\ell}}\right).$$

We can notice also that, thanks to the symmetries of the von Mises functional G defined as in (3.2), it is not necessary to compute all the $\left|\mathcal{B}_{\frac{m}{2},N}\right|$ terms of the sums in the formulae (3.4) and (3.5). Instead, the computation of $\mathcal{N}(G,N) = \binom{\frac{N(N+1)+m}{2}-1}{\frac{m}{2}}$ terms is sufficient, provided that they are counted with their respective multiplicity.

3.2. A showcase. As numerical application we study the following shape optimization problem

$$\left|\begin{array}{l} \text{Find } \Omega_{\mathrm{opt}} \in \mathcal{S}_{adm} \text{ minimizing } \Omega \mapsto \mathrm{Vol}(\Omega), \\ \text{where, for all } \omega \in \mathcal{O}, \text{ the state } \mathbf{u} \in \mathrm{H}^1(\Omega)^3 \text{ solves:} \\ \qquad \left\{\begin{array}{rcll} -\mathrm{div}\ \mathbf{A}\nabla\mathbf{u}(\omega) & = & \mathbf{0} & \text{in } \Omega, \\ \mathbf{A}\nabla\mathbf{u}(\omega)\,\mathbf{n} & = & \mathbf{g}(\omega) & \text{on } \Gamma_{\mathrm{N}}, \\ \mathbf{A}\nabla\mathbf{u}(\omega)\,\mathbf{n} & = & \mathbf{0} & \text{on } \Gamma_0, \\ \mathbf{u}(\omega) & = & \mathbf{0} & \text{on } \Gamma_{\mathrm{D}}. \end{array}\right. \\ \text{with the constraint:} \quad \mathbb{E}\left[\mathcal{G}_6(\Omega,\mathbf{g})\right] \leq M_0^6, \end{array}\right. \tag{3.7}$$

where $M_0 > 0$ is a given upper bound for the constraint functional $\mathbb{E}\left[\mathcal{G}_6(\Omega,\mathbf{g})\right]$.

The structure to be optimized is a cylinder-like shape with axis $z = 0$, reported in Figure 1. Dirichlet boundary conditions are imposed on a thin stripe on the lateral surface, while the random load $\mathbf{g}$ is applied on a ring-shaped section on the upper surface of the structure. We consider the mechanical load $\mathbf{g} \in \mathrm{L}^6\left(\mathcal{O},\mathbb{P};\mathrm{L}^2\left(\Gamma_{\mathrm{N}}\right)\right)$ to have the following structure

$$\mathbf{g}(\omega) = \mathbf{g}_1\, X_1(\omega) + \mathbf{g}_2\, X_2(\omega) \qquad \text{for almost all event } \omega \in \mathcal{O}.$$

The loads $\mathbf{g}_1$ and $\mathbf{g}_2$ are set as constant vectors on Γ_{N}, parallel to the axes x and y respectively, thus tangent to the surface. Moreover, we consider the random variables X_1 and X_2 to follow centered Gaussian distributions with variance σ_1 and σ_2 respectively.

Remark 3.3. Unfortunately, the hypothesis (2.6) about the separation of the Dirichlet and Neumann boundaries cannot be verified in most practical situations. Indeed, the regularity of the displacement $\mathbf{u}$ is limited by the possible appearance of a finite number of singularities around the junctions of the two portions of the

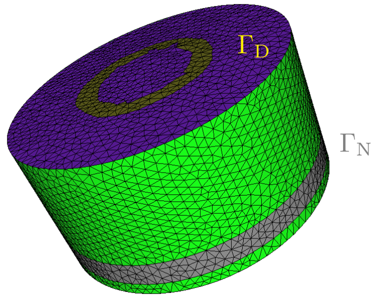
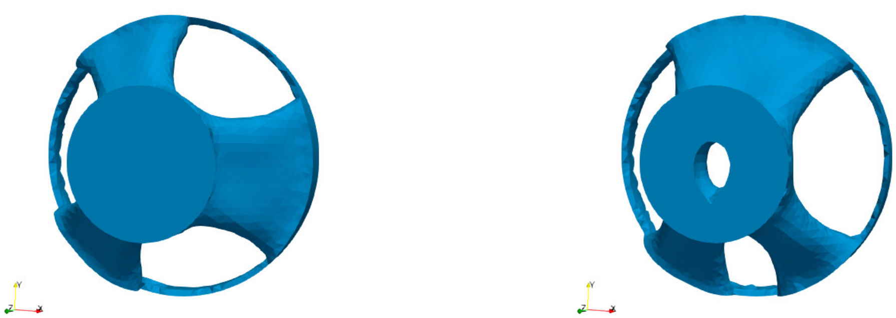


FIG. 1. *Representation of the structure to be optimized. The surface Γ_{D} is the thin grey stripe on the lateral surface, while Γ_{N} is the ring-shaped portion of the upper surface marked in yellow.*

boundary where natural or essential conditions are imposed [12]. In this section we will not focus on the study of the compatibility conditions to avoid the emergence of singularities, but we present the results of some simulations where no difficulty related to the regularity of the solution has been observed.

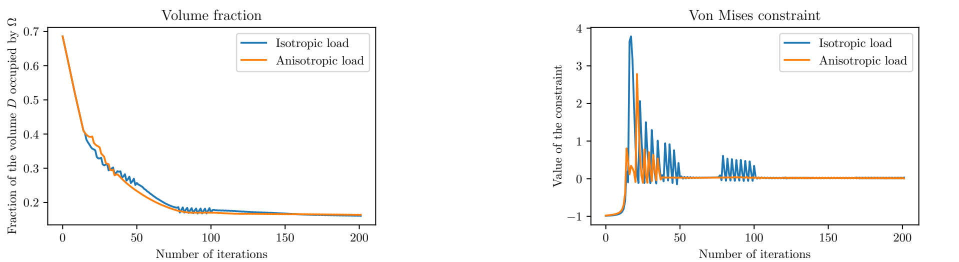
From the numerical point of view, we represent the structure by using a level-set function on a fixed mesh $\mathcal{T}_h$ covering a fixed domain D containing every admissible shape in $\mathcal{S}_{adm}$ (see [7, 5] for further information on the level-set approach to shape optimization). The linear elasticity equations (2.4) and the adjoint problems (2.20) are defined on the entire domain $D = \Omega \cup \Omega^C$, by using an *ersatz material* approximation in Ω^C to assure the well-posedness of the problems (see [7, 22]). The elasticity and adjoint equations are solved by using the *FreeFem++* environment [30].



(a) Isotropic mechanical load (b) Anisotropic mechanical load

FIG. 2. *Optimal shapes found by the nullspace optimization algorithm.*

From the observation of Figure 2 and Figure 3 we remark firstly the efficiency of the *nullspace optimization* algorithm in the solution of the constrained optimization problem (3.7). Indeed, the value of the objective functional is decreasing (see Figure 3a). As seen in Figure 3b, the constraint on the expectation of $\mathcal{G}_6$ is saturated in less than 50 iterations for the anisotropic case. In the isotropic case, we observe some oscillations in the constraint saturation around iteration 80, which are due to



(a) [Convergence of the volume of Ω, as the objective

(b) Convergence of the constraint $\mathbb{E}\left[\mathcal{G}_6(\Omega, \mathbf{g})\right] - M_0^6$

FIG. 3. *Convergence of the objective and constraint of problem* (3.7).

	Isotropic case	**Anisotropic case**
Final volumic fraction $\mathrm{Vol}(\Omega_{\mathrm{opt}})/\,\mathrm{Vol}(D)$	0.1608	0.164
Normalized saturation of the constraint $(\mathbb{E}\left[\mathcal{G}_6\right] - M_0^6)/M_0^6$	0.03002	0.005351
Execution time	129 minutes	148 minutes

TABLE 1

Numerical results of the solution of problem (3.7) *for an isotropic and anisotropic mechanical load.*

a change in the topology around that step of the optimization. The shapes of Figure 2 show that a ramified structure presents the minimal volume ensuring enough resistance with respect to random mechanical loads. Moreover, if the direction of the mechanical load $\mathbf{g}(\cdot)$ is not uniformly distributed in the interval $[0, 2\pi]$, the branches tend to align parallel to the most probable direction of the load (see Figure 2b).

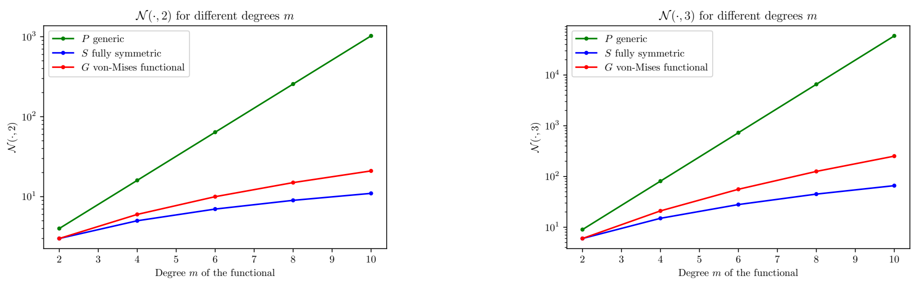
Finally, we remark that the constraint imposed in problem (3.7) is a quite conservative estimate for the expected value of the L^6-norm of the von Mises stress. Thanks to the inequality (3.3) and the fact that the optimal shapes respect the constraint $\mathbb{E}\left[\mathcal{G}_6\right] \leq M_0^6$, we deduce that the average of the L^6-norm of the von Mises stress in the structures is actually less than the chosen threshold M_0.

4. Conclusions and perspectives. This article studied a procedure of shape optimization of polynomial functionals, where the external load applied to the structure is subject to uncertainties. Particular attention has been payed to the optimization of linear elastic structures, and we adopted the level-set approach to topology optimization. The present work proposed an extension of the technique proposed in [18] to the case of continuous multilinear functionals, and relies on the linearization properties of the tensor product between elements of a Banach space.

A significant obstacle in the application of this method is the number of terms appearing in the sums of (2.18) (for the computation of the functional of interest), and (2.19) (for its derivative). Let us recall the definition of $\mathcal{N}(P, N)$ introduced at the end of subsection 2.3 as the minimal number of terms that are necessary to compute $\mathbb{E}\left[P(\mathbf{u}, \ldots, \mathbf{u})\right]$ and its derivative, where P is a m-multilinear functional, and $\mathbf{u}$ is described by N random variables. Let us consider three different bounded m-multilinear functionals: a generic functional P, a functional S which is completely symmetric in its arguments, and the von Mises functional G defined in (3.2). We recall that in subsection 2.3 and in subsection 3.1 we found the following expressions for the number of terms necessary to compute the expectations of such functionals

$$\mathcal{N}(P,N) = N^m, \quad \mathcal{N}(S,N) = \binom{N+m-1}{m} \text{ and } \mathcal{N}(G,N) = \binom{\frac{N(N+1)}{2} + \frac{m}{2} - 1}{\frac{m}{2}}.$$

As represented in Figure 4, the number of terms to be computed increases rapidly with the degree m of the multilinear functional, even if the number of random variables N is limited to 2 or 3. Naturally, the presence of symmetries in the multilinear mapping greatly reduces the number of terms to be computed, but the problem can still become too complex if the degree m required is too high.



(a) Case of $N = 2$ random variables (b) Case of $N = 3$ random variables

FIG. 4. *Evaluation of $\mathcal{N}(P,N)$, $\mathcal{N}(S,N)$, and $\mathcal{N}(G,N)$ for different degrees m of the functionals examined.*

As remedy to this issue, we suggest to exploit the symmetric nature of the correlation tensor, and study the application of some techniques of tensor decomposition. One promising solution consists in the approximation of the discretized correlation tensor as a sum of tensor of rank 1, by using the CP-decomposition. This technique and other kinds of tensor decompositions are detailed in [36, 35, 16], and have been implemented in Python libraries as TensorLy [37]. However, its interpretation and applicability in the field of shape optimization are still to be investigated.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Elsevier, June 2003.

[2] G. Allaire, *Conception optimale de structures*, vol. 58 of Mathématiques & applications, Springer Berlin Heidelberg, Berlin, 2007.

[3] G. Allaire and C. Dapogny, *A deterministic approximation method in shape optimization under random uncertainties*, SMAI Journal of Computational Mathematics, 1 (2015), pp. 83–143. Publisher: Société de Mathématiques Appliquées et Industrielles (SMAI).

[4] G. Allaire, C. Dapogny, and P. Frey, *Shape optimization with a level set based mesh evolution method*, Computer Methods in Applied Mechanics and Engineering, 282 (2014), pp. 22–53.

[5] G. Allaire, C. Dapogny, and F. Jouve, *Shape and topology optimization*, vol. 22, Elsevier, 2021.

[6] G. Allaire and F. Jouve, *Minimum stress optimal design with the level set method*, Engineering Analysis with Boundary Elements, 32 (2008), pp. 909–918.

[7] G. Allaire, F. Jouve, and A.-M. Toader, *Structural optimization using sensitivity analysis and a level-set method*, Journal of Computational Physics, 194 (2004), pp. 363–393.

[8] G. Allaire and O. Pantz, *Structural optimization with FreeFem++*, Structural and Multidisciplinary Optimization, 32 (2006), pp. 173–181.

[9] S. Amstutz and A. A. Novotny, *Topological optimization of structures subject to Von Mises stress constraints*, Structural and Multidisciplinary Optimization, 41 (2010), pp. 407–420.
[10] M. P. Bendsøe, *Optimal shape design as a material distribution problem*, Structural optimization, 1 (1989), pp. 193–202.
[11] M. P. Bendsøe and O. Sigmund, *Topology optimization. Theory, methods, and applications. 2nd ed., corrected printing*, Springer-Verlag, Jan. 2004.
[12] R. Bey, J.-P. Lohéac, and M. Moussaoui, *Singularities of the solution of a mixed problem for a general second order elliptic equation and boundary stabilizationof the wave equation*, Journal de Mathématiques Pures et Appliquées, 78 (1999), pp. 1043–1067.
[13] D. Bump, *Lie Groups*, vol. 225 of Graduate Texts in Mathematics, Springer, New York, NY, 2004.
[14] S. Chen and W. Chen, *A new level-set based approach to shape and topology optimization under geometric uncertainty*, Structural and Multidisciplinary Optimization, 44 (2011), pp. 1–18.
[15] S. Chen, W. Chen, and S. Lee, *Level set based robust shape and topology optimization under random field uncertainties*, Structural and Multidisciplinary Optimization, 41 (2010), pp. 507–524.
[16] P. Comon, X. Luciani, and A. L. F. de Almeida, *Tensor decompositions, alternating least squares and other tales*, Journal of Chemometrics, 23 (2009), pp. 393–405. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cem.1236.
[17] J. Céa, *Conception optimale ou identification de formes, calcul rapide de la dérivée directionnelle de la fonction coût*, ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, 20 (1986), pp. 371–402.
[18] M. Dambrine, C. Dapogny, and H. Harbrecht, *Shape Optimization for Quadratic Functionals and States with Random Right-Hand Sides*, SIAM Journal on Control and Optimization, 53 (2015), pp. 3081–3103.
[19] M. Dambrine, H. Harbrecht, and B. Puig, *Computing quantities of interest for random domains with second order shape sensitivity analysis*, ESAIM: Mathematical Modelling and Numerical Analysis, 49 (2015), pp. 1285–1302. Number: 5 Publisher: EDP Sciences.
[20] M. Dambrine, H. Harbrecht, and B. Puig, *Incorporating knowledge on the measurement noise in electrical impedance tomography*, ESAIM: Control, Optimisation and Calculus of Variations, 25 (2019), p. 84. Publisher: EDP Sciences.
[21] M. Dambrine and V. Karnaev, *Robust obstacle reconstruction in an elastic medium*, Discrete and Continuous Dynamical Systems - B, (2023), pp. 0–0. Publisher: Discrete and Continuous Dynamical Systems - B.
[22] M. Dambrine and D. Kateb, *On the Ersatz material approximation in level-set methods*, ESAIM: Control, Optimisation and Calculus of Variations, 16 (2010), pp. 618–634.
[23] C. Dapogny, C. Dobrzynski, and P. Frey, *Three-dimensional adaptive domain remeshing, implicit domain meshing, and applications to free and moving boundary problems*, Journal of Computational Physics, 262 (2014), pp. 358–378.
[24] C. Dapogny and P. Frey, *Computation of the signed distance function to a discrete contour on adapted triangulation*, Calcolo, 49 (2012), pp. 193–219.
[25] F. de Gournay, *Velocity Extension for the Level-set Method and Multiple Eigenvalues in Shape Optimization*, SIAM Journal on Control and Optimization, 45 (2006), pp. 343–367.
[26] P. D. Dunning and H. A. Kim, *Robust Topology Optimization: Minimization of Expected and Variance of Compliance*, AIAA Journal, 51 (2013), pp. 2656–2664. Publisher: American Institute of Aeronautics and Astronautics _eprint: https://doi.org/10.2514/1.J052183.
[27] F. Feppon, *Optimisation topologique de systèmes multiphysiques*, PhD thesis, Université Paris Saclay (COmUE), Dec. 2019.
[28] F. Feppon, G. Allaire, and C. Dapogny, *Null space gradient flows for constrained optimization with applications to shape optimization*, ESAIM: Control, Optimisation and Calculus of Variations, 26 (2020), p. 90. Publisher: EDP Sciences.
[29] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela, *Hölder Continuity and Optimal Control for Nonsmooth Elliptic Problems*, Applied Mathematics and Optimization, 60 (2009), pp. 397–428.
[30] F. Hecht, *New development in freefem++*, Journal of Numerical Mathematics, 20 (2012), pp. 1–14. Publisher: De Gruyter.
[31] A. Henrot and M. Pierre, *Shape variation and optimization: a geometrical analysis*, no. 28 in Tracts in Mathematics, European Mathematical Society, Zurich, 2018.
[32] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer, Cham, 2016.

[33] R. M. Jones, *Deformation Theory of Plasticity*, Bull Ridge Corporation, Blacksburg, Virginia, 2008.

[34] R. V. Kadison and J. R. Ringrose, *Elementary Theory: Fundamentals of the Theory of Operator Algebras*, vol. I, Elsevier, New York, 1983.

[35] T. G. Kolda, *Numerical optimization for symmetric tensor decomposition*, Mathematical Programming, 151 (2015), pp. 225–248.

[36] T. G. Kolda and B. W. Bader, *Tensor Decompositions and Applications*, SIAM Review, 51 (2009), pp. 455–500.

[37] J. Kossaifi, Y. Panagakis, A. Anandkumar, and M. Pantic, *TensorLy: Tensor Learning in Python*, Journal of Machine Learning Research, 20 (2019), pp. 1–6.

[38] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer, Berlin, Heidelberg, 1972.

[39] M. Reed and B. Simon, eds., *Methods of Modern Mathematical Physics, Volume I: Functional Analysis*, Academic Press, Jan. 1972.

[40] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer, London, 2002.

[41] S. Salsa, *Partial Differential Equations in Action: From Modeling To Theory*, UNITEXT, Springer International Publishing AG, Cham, 3 ed., Apr. 2015.

[42] C. Schwab and R. Todor, *Sparse Finite Elements for Stochastic Elliptic Problems – Higher Order Moments*, Computing, 71 (2003), pp. 43–63.

[43] C. Schwab and R.-A. Todor, *Sparse finite elements for elliptic problems with stochastic loading*, Numerische Mathematik, 95 (2003), pp. 707–734.

[44] O. Sigmund and K. Maute, *Topology optimization approaches*, Structural and Multidisciplinary Optimization, 48 (2013), pp. 1031–1055.

[45] J. Simon and F. Murat, *Sur le contrôle par un domaine géométrique*, Laboratoire d'Analyse Numérique de l'Université de Paris VI, (1976).

[46] W. S. Slaughter, *The linearized theory of elasticity*, Birkhäuser, Boston, 2002.

[47] N. P. van Dijk, K. Maute, M. Langelaar, and F. van Keulen, *Level-set methods for structural topology optimization: a review*, Structural and Multidisciplinary Optimization, 48 (2013), pp. 437–472.

Appendix A. Mathematical setting and tools.

A.1. Shape optimization. Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary, in dimension $d = 2$ or 3. If $\boldsymbol{\theta} \in \mathrm{W}^{1,\infty}\left(\mathbb{R}^d\right)^d$ is a Lipschitz continuous vector field such that $\|\boldsymbol{\theta}\|_{1,\infty} = \|\boldsymbol{\theta}\|_{\infty} + \|\nabla\boldsymbol{\theta}\|_{\infty} < 1$, we define the deformed domain $\Omega_{\boldsymbol{\theta}}$ as $\Omega_{\boldsymbol{\theta}} = (\mathbb{I} + \boldsymbol{\theta})\,\Omega$. For the sake of simplicity, we consider a class of admissible shapes $\mathcal{S}_{adm}$, and a class Θ_{adm} of vector fields such that, for all $\boldsymbol{\theta} \in \Theta_{adm}$, the deformed set $\Omega_{\boldsymbol{\theta}}$ belongs to $\mathcal{S}_{adm}$. Let $J(\cdot)$ be a shape functional $J : \mathcal{S}_{adm} \to \mathbb{R}$. At first, we recall the notion of shape differentiability, as introduced in [31, Chapter 5] or in [2, Section 6.3].

Definition A.1 (Fréchet differentiable shape functional). *A shape functional $J : \mathcal{S}_{adm} \to \mathbb{R}$ is shape differentiable according to Fréchet at Ω if there exists a linear continuous function $A_\Omega : \mathrm{W}^{1,\infty}\left(\mathbb{R}^d\right)^d \to \mathbb{R}$ such that*

$$J(\Omega_{\boldsymbol{\theta}}) = J(\Omega) + A_\Omega(\boldsymbol{\theta}) + o(\boldsymbol{\theta})$$

for all $\boldsymbol{\theta} \in \mathrm{W}^{1,\infty}\left(\mathbb{R}^d\right)^d$, where $\lim\limits_{\boldsymbol{\theta}\to 0} \frac{o(\boldsymbol{\theta})}{\|\boldsymbol{\theta}\|_{1,\infty}} = 0$. The linear form A_Ω is called shape derivative of J in Ω and is denoted as $\frac{\mathrm{d}}{\mathrm{d}\Omega}J(\boldsymbol{\theta})$.

If the domain Ω is sufficiently regular, we can assume that the value of the derivative $J'(\Omega)(\boldsymbol{\theta})$ depends only on the normal component of the vector field $\boldsymbol{\theta}$ on the surface $\partial\Omega$ of the domain. Such result derives from the following *structure theorem*, proven by Hadamard and stated as [31, Proposition 5.9.1].

Theorem A.2 (Structure theorem). *Let $\Omega \in \mathcal{S}_{adm}$ be a $\mathcal{C}^1$ domain, and let us denote by $\mathbf{n}(\mathbf{s})$ the vector normal to the surface $\partial\Omega$ in $\mathbf{s} \in \partial\Omega$. We suppose that $J : \mathcal{S}_{adm} \to \mathbb{R}$ is a differentiable functional. If $(\boldsymbol{\theta}\cdot\mathbf{n}) = 0$ on the entire surface $\partial\Omega$, then $J'(\Omega)(\boldsymbol{\theta}) = 0$.*

In the context of shape optimization, the shape derivative is used to identify a direction of deformation $\boldsymbol{\theta}_{\mathrm{def}}$ such that $J'(\Omega)(\boldsymbol{\theta}_{\mathrm{def}}) < 0$, which acts as direction of descent in a suitable gradient-based optimization algorithm (see e.g. [8, 28, 5]).

A.2. Tensor product in Hilbert spaces. In [43, 18] tensor products between Hilbert spaces have been used to work with the stochastic moment of order 2 of random quantities. In [42] the same technique has been extended to treat the stochastic moment of order $m > 2$. Here, we recall the main definitions and results about the tensor product in Hilbert spaces as presented in [39, 40, 43, 42, 18].

Definition A.3 (Tensor product in vector spaces). *For a positive integer $m \geq 2$, let us consider the vector spaces $\mathcal{X}_1, \ldots, \mathcal{X}_m$. We denote $\hat{\mathfrak{P}}_m(\mathcal{X}_1, \ldots, \mathcal{X}_m)$ the space of all m-multilinear forms on $\prod_{i=1}^m \mathcal{X}_i$. For $(\mathcal{X}_1, \ldots, \mathcal{X}_m) \in \prod_{i=1}^m \mathcal{X}_i$, the tensor product $x_1 \otimes \ldots \otimes x_m$, also written as $\bigotimes_{i=1}^m x_i$, is a real valued linear application defined on $\hat{\mathfrak{P}}_m(\mathcal{X}_1, \ldots, \mathcal{X}_m)$ such that, for all $P_m \in \hat{\mathfrak{P}}_m(\mathcal{X}_1, \ldots, \mathcal{X}_m)$,*

$$\left(\bigotimes_{i=1}^m x_i\right)(P_m) = P_m(\mathcal{X}_1, \ldots, \mathcal{X}_m).$$

If all spaces $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are Hilbert spaces, it is possible to define a product space with a Hilbertian structure.

Definition A.4 (Tensor product between Hilbert spaces). *Let $\left(\mathcal{H}_i, \langle\cdot,\cdot\rangle_{\mathcal{H}_i}\right)$ be m*

Hilbert spaces, with $m \geq 2$. We define the set $\mathcal{V}$ as

$$\mathcal{V} = \operatorname{span}\left\{ \bigotimes_{i=1}^{m} x_i \quad \textit{such that} \quad x_i \in \mathcal{H}_i \quad \forall i = 1 \dots m \right\}.$$

Let $\langle \cdot, \cdot \rangle_{\otimes} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear operation such that

$$\left\langle \bigotimes_{i=1}^{m} x_i, \bigotimes_{i=1}^{m} y_i \right\rangle_{\otimes} = \prod_{i=1}^{m} \langle x_i, y_i \rangle_{\mathcal{H}_i}. \tag{A.1}$$

for any choice of $x_i, y_i \in \mathcal{H}_i$. We denote $\|\cdot\|_{\otimes}$ the norm induced by the tensor product $\langle \cdot, \cdot \rangle_{\otimes}$.

The operation introduced in (A.1) is an inner product in $\mathcal{V}$ (see [39, Section II.4]). The tensor product of the Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_m$ is the completion of $\mathcal{V}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\otimes}$, and is denoted $\bigotimes_{i=1}^{m} \mathcal{H}_i$. If all the m Hilbert spaces coincide with a single Hilbert space $\mathcal{H}$, we denote their tensor product as $\mathcal{H}^{\otimes m}$.

Definition A.5 (Operator norm). *For any real valued linear operator P defined on a normed vector space $\mathcal{X}$, we denote its operator norm as:*

$$\|P\|_{\mathrm{OP}} = \sup_{\|x\|_{\mathcal{X}}=1} |P(x)|.$$

Similarly, for any m-multilinear functional $P_m : \mathcal{X}_1, \dots, \mathcal{X}_m \to \mathbb{R}$ its operator norm is defined as:

$$\|P_m\|_{\mathrm{OP}} = \sup_{\|x_i\|_{\mathcal{X}_i}=1\, \forall i} |P_m(\mathcal{X}_1, \dots, \mathcal{X}_m)|.$$

As stated in [40, section 1.2], a primary purpose of the tensor product is the of multilinear mappings into linear ones.

Proposition A.6 (Linearization of bounded multilinear functionals). *Let us consider a real-valued, bounded, multilinear functional $P : \prod_{i=1}^{m} \mathcal{H}_i \to \mathbb{R}$ defined on the separable Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_m$. Then, there exists a unique linear functional $\widehat{P}_m : \bigotimes_{i=1}^{m} \mathcal{H}_i \to \mathbb{R}$ such that $\widehat{P}_m$ is continuous, and for all $(x_1, \dots, x_m) \in \prod_{i=1}^{m} \mathcal{H}_i$, $\widehat{P}_m \left(\bigotimes_{i=1}^{m} x_i\right) = P(x_1, \dots, x_m)$.*

The existence of the functional $\widehat{P}_m$ for any m-multilinear continuous mapping P is often referred as the *universal property of the tensor product* [13, Chapter 9] and is proven in [34, Theorem 2.6.4].

A.3. Modeling of the uncertainties. In order to model the uncertainties, we use the formalism of Bochner spaces, which extends the theory of integration to Banach-valued functions [32, Chapter 1].

We recall the definition of measurable and integrable functions in the context of Bochner spaces for a generic measure μ on the σ-algebra $\mathcal{F}$.

Definition A.7 (μ-simple and strongly μ-measurable functions). *A function $g : \mathcal{O} \to \mathcal{X}$ is said to be μ-simple if it can be written in the form*

$$\sum_{i=1}^{N} \chi_{A_i} \mathbf{x}_i,$$

where N is a finite positive integer, $\mathbf{x}_i \in \mathcal{X}$, $A_i \in \mathcal{F}$, and $\mu(A_i) < \infty$ for all $i \in \{1, \dots, N\}$, and χ_A is the characteristic function for the set A.

A function $f : \mathcal{O} \to \mathcal{X}$ is said to be strongly μ-measurable if there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of μ-simple functions converging to f μ-almost everywhere.

Definition A.8 (Bochner integral). *The Bochner integral of a simple function $g : \mathcal{O} \to \mathcal{X}$ sugh that $g = \sum_{i=1}^{N} \chi_{A_i} \mathbf{x}_i$ with respect to the measure μ is defined by*

$$\int_{\mathcal{O}} g \,\mathrm{d}\mu = \sum_{i=1}^{N} \mu(A_i)\, \mathbf{x}_i \quad \in \mathcal{X}.$$

A strongly μ-measurable function f is Bochner integrable with respect to the measure μ if there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of μ-simple functions $g_i : \mathcal{O} \to \mathcal{X}$ such that

$$\lim_{i \to \infty} \int_{\mathcal{O}} \|f - g_i\|_{\mathcal{X}} \,\mathrm{d}\mu = 0,$$

where the (real) integral is intended at the sense of Lebesgue. The Bochner integral of such a Bochner integrable function function is defined as

$$\int_{\mathcal{O}} f \,\mathrm{d}\mu = \lim_{i \to \infty} \int_{\mathcal{O}} g_i \,\mathrm{d}\mu \quad \in \mathcal{X}.$$

Moreover, the value of $\int_{\mathcal{O}} f$ is independent from the choice of the sequence $\{g_i\}_{i=1}^{\infty}$.

Once defined the integration for Banach-valued functions, we can introduce the Bochner spaces as the equivalent of the usual L^p spaces for real-valued functions.

Definition A.9 (Bochner spaces and equivalence). *A Bochner integrable function $f : \mathcal{O} \to \mathcal{X}$ belongs to the space $\mathcal{L}^p(\mathcal{O}, \mu; \mathcal{X})$ for $i \le p < \infty$ if and only if $\int_{\mathcal{O}} \|f\|_{\mathcal{X}}^p \,\mathrm{d}\mu < \infty$.*

A Bochner integrable function $f : \mathcal{O} \to \mathcal{X}$ belongs to the space $\mathcal{L}^\infty(\mathcal{O}, \mu; \mathcal{X})$ if and only if there exist a real positive number $r < \infty$ such that $\mu\left(\{\Omega \in \mathcal{O} \,:\, \|f\|_{\mathcal{X}} \ge r\}\right) = 0$.

Two strongly μ-measurable function f and g are said to be equivalent if the subset of $\mathcal{O}$ where f is different from g has measure 0. The equivalence relation is denoted as $f \sim g$.

The Bochner space $\mathrm{L}^p\left(\mathcal{O}, \mu; \mathcal{X}\right)$ for $1 \le p \le \infty$ is defined as the quotient of $\mathcal{L}^p(\mathcal{O}, \mu; \mathcal{X})$ with respect to the equivalence relation "$\sim$".

Bochner spaces are also Banach spaces with respect to the following norms:

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathcal{O}} \|f\|_{\mathcal{X}}^p \,\mathrm{d}\mu \right)^{1/p} && \text{for} \quad 1 \le p < \infty; \\ \|f\|_\infty &= \inf\left\{ r \ge 0 \,:\, \mu\left(\{\Omega \in \mathcal{O} \,:\, \|f\|_{\mathcal{X}} \ge r\}\right) = 0 \right\}. \end{aligned}$$

Having stated the main definition about generic Bochner spaces, let us focus on the case where we consider a probability measure $\mathbb{P}$. At first, we can remark the following embedding of Bochner spaces.

Proposition A.10 (Embeddings in Bochner spaces). *Let ℓ and m be real numbers such that $1 \le \ell < m < \infty$. Then, the following inclusion is true:*

$$\mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathcal{X}\right) \subset \mathrm{L}^\ell\left(\mathcal{O}, \mathbb{P}; \mathcal{X}\right).$$

In particular, if $f \in \mathrm{L}^m\left(\mathcal{O}, \mathbb{P}; \mathcal{X}\right)$, then f belongs also to $\mathrm{L}^1\left(\mathcal{O}, \mathbb{P}; \mathcal{X}\right)$.

Proof. The proof relies simply on Hölder's inequality [41, Equation (1.9)]. Let us denote $p = \frac{m}{l}$ and q its conjugate such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have:

$$\int_{\mathcal{O}} \|f\|_{\mathcal{X}}^{l} \, \mathrm{d}\mu = \int_{\mathcal{O}} \|f\|_{\mathcal{X}}^{l} \, 1 \, \mathrm{d}\mu = \left(\int_{\mathcal{O}} \|f\|_{\mathcal{X}}^{l\frac{m}{l}} \, \mathrm{d}\mu \right)^{1/p} \left(\int_{\mathcal{O}} 1 \right)^{1/q} = \|f\|_{\mathrm{L}^m(\mathcal{O},\mathbb{P};\mathcal{X})}^{l} < \infty.$$

∎

We recall the definition of the expectation operator in Bochner spaces and a classical result about the commutation of the expectation and a closed linear operator.

Definition A.11 (Expectation). *The expectation operator* $\mathbb{E}[\,\cdot\,] : \mathrm{L}^1(\mathcal{O}, \mathbb{P}; \mathcal{X}) \to \mathcal{X}$ *is the bounded linear operator such that, for all* $f \in \mathrm{L}^1(\mathcal{O}, \mathbb{P}; \mathcal{X})$,

$$\mathbb{E}[f] = \int_{\mathcal{O}} f \, \mathrm{d}\mathbb{P} \quad \in \mathcal{X}.$$

Proposition 1.2.3 and Equation (1.2) of [32] allow to state the following result on continuous operators on Bochner spaces.

Proposition A.12. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces,* $f : \mathcal{O} \to \mathcal{X}$ *be a Bochner-integrable function, and* $T : \mathcal{X} \to \mathcal{Y}$ *be a continuous linear operator. Then,* $T f : \mathcal{O} \to \mathcal{Y}$ *is a Bochner-integrable function, and*

$$\mathbb{E}[T f] = T\, \mathbb{E}[f].$$

A more general version of Proposition A.12 is known in literature as Hille's theorem (see [32, Theorem 1.2.4]), which does not require T to be a continuous operator, but only a closed one on a subspace of $\mathcal{X}$.

However, Proposition 1.2.3 and Equation (1.2) of [32] point out that, for continuous operators, it is not necessary to prove Hille's theorem to get the same properties, since they descend directly from the definition of Bochner integral.

Appendix B. On the numerical implementation.

The optimization algorithm chosen to solve problem (3.7) is the *nullspace optimization* algorithm, introduced in [28]. This algorithm requires the computation of the shape derivatives of the objective functional as well as of the constraints, motivating the application of the formula introduced in (3.5) for the derivative of $\mathbb{E}[\mathcal{G}_6(\Omega, \mathbf{g})]$. The *nullspace optimization* algorithm is implemented in python. The packages *pyfreefem*[1] and *pymedit*[2] [27] have been used to interface the python general framework with the *FreeFem++* finite-element solver and the methods for the computation of the signed-distance function [24] and advection of the level-set [23] provided in the ISCD toolbox[3].

The numerical results of two different simulations are discussed: in the first case we consider the random variables X_1 and X_2 to have an identical distribution (isotropic distribution of the external mechanical load), while the second case considers an asymmetry in the variances of the two random variables (anisotropic distribution). The parameters used in the simulation are reported in Table 2. The shapes obtained by the execution of 200 iterations of the *nullspace optimization* algorithm for both cases are reported in Figure 2, and the convergences of the objective and the constraint functions in Figure 3. All simulations have been performed on a Virtualbox virtual

[1] Available online at https://pypi.org/project/pyfreefem/
[2] Available online at https://pypi.org/project/pymedit/
[3] Available online at https://github.com/ISCDtoolbox

machine Linux with 1GB of dedicated memory, installed on a Dell PC equipped with a 2.80 GHz Intel i7 processor. The numerical results are reported in Table 1.

Heigth of the domain D	12.0	
Radius of the cylinder D	12.0	
Region Γ_{N}		
inner radius	4.0	
outer radius	6.0	
Region Γ_{D}		
thickness	2.0	
distance from the edge of D	1.0	
Mesh size parameters		
minimal element size `hmin`	0.4	
maximal element size `hmax`	0.8	
gradation value `hgrad`	1.3	
Elastic coefficients		
Young's modulus E	15	
Poisson's ration ν	0.35	
Ersatz material coefficient $\varepsilon_{\mathrm{ers}}$	10^{-3}	
Treshold M_0	3.0	
Variances of the random variables	isotropic	anisotropic
σ_1^2	2.5	1.0
σ_2^2	2.5	4.0
Number of iterations	200	

TABLE 2

Numerical parameters for problem (3.7) for the cases of random variables with equal and with different variances.